Theory of Computer Science

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Exercise Sheet 1

Due: Sunday, February 26, 2017

Note: Submissions that are exclusively created with LaTeX will receive a bonus mark. Please submit only the resulting PDF file (or a printout of this file).

Note: The goal of this exercise is to learn how to correctly express formal proofs. A formally correct proof consists of single steps where each step follows *immediately* from the previous steps or from the assumptions (for example when replacing a value by its definition). Please write down your proofs in detail and in a formal fashion. Examples can be fould in the lecture slides.

Exercise 1.1 (Direct Proof; 2 + 2 marks)

(a) Prove the following statement for any sets A, B, and C with a direct proof:

$$((A \cap B) \cap C) = (A \cap (B \cap C))$$

(b) Refute the claim that the following statement holds for any sets A, B, and C by specifying a *counter example* (i.e., sets A, B, and C) and showing that it does not satisfy the statement.

$$((A \setminus B) \setminus C) = (A \setminus (B \setminus C))$$

Exercise 1.2 (Proof by Contradiction; 3 marks)

Prove by contradiction that for all $n \in \mathbb{N}_0$ the following holds: if $n + 13$ is prime, then n is not prime.

Hint: 2 is the only even prime number.

Exercise 1.3 (Structural Induction; 2 + 3 marks)

(a) We inductively define a set of simple mathematical expressions which only utilize the following symbols: "Z", "T", "$\oplus$", "$\otimes$", "$[\![$", and "$]\!]$". The set $\mathcal{E}$ of *simple expressions* is inductively defined as follows:

- Z and T are simple expressions.
- If x and y are simple expressions, $[\![x \otimes y]\!]$ is also a simple expression.
- If x and y are simple expressions, $[\![x \oplus y]\!]$ is also a simple expression.

Examples for simple expressions: T, $[\![\mathtt{T} \otimes \mathtt{Z}]\!]$, $[\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]$

Furthermore we define a function $f : \mathcal{E} \to \mathbb{N}_0$ as follows:

- $f(\mathtt{Z}) = 0$, $f(\mathtt{T}) = 2$
- $f([\![x \otimes y]\!]) = f(x) \cdot f(y)$
- $f([\![x \oplus y]\!]) = f(x) + f(y)$

So for example: $f(\mathtt{T}) = 2,\ f([\![\mathtt{T} \otimes \mathtt{Z}]\!]) = f(\mathtt{T}) \cdot f(\mathtt{Z}) = 2 \cdot 0 = 0,\ f([\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]) = 6.$

Prove the following property for all simple expressions $x \in \mathcal{E}$ by structural induction:

$$f(x) \text{ is even.}$$

(b) We define two functions over binary trees (as presented in the lecture): $\mathit{height} : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to its height and $\mathit{leaves} : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to the number of its leaves.

- $\mathit{height}(\square) = 1$
- $\mathit{height}(\langle B_L, \bigcirc, B_R \rangle) = \max(\mathit{height}(B_L), \mathit{height}(B_R)) + 1$
- $\mathit{leaves}(\square) = 1$
- $\mathit{leaves}(\langle B_L, \bigcirc, B_R \rangle) = \mathit{leaves}(B_L) + \mathit{leaves}(B_R)$

Prove the following property for all binary trees $B \in \mathcal{B}$ by structural induction:

$$\mathit{leaves}(B) \leq 2^{\mathit{height}(B)-1}.$$