

# Foundations of Artificial Intelligence

## 28. Constraint Satisfaction Problems: Decomposition Methods

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April 19, 2017 — 28. Constraint Satisfaction Problems:  
Decomposition Methods

## 28.1 Decomposition Methods

## 28.2 Conditioning

## 28.3 Tree Decomposition

## 28.4 Summary

## Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- ▶ 22.–23. Introduction
- ▶ 24.–26. Basic Algorithms
- ▶ 27.–28. Problem Structure
  - ▶ 27. Constraint Graphs
  - ▶ 28. Decomposition Methods

## 28.1 Decomposition Methods

## More Complex Graphs

What if the constraint graph is not a tree and does not decompose into several components?

- ▶ idea 1: **conditioning**
- ▶ idea 2: **tree decomposition**

German: Konditionierung, Baumzerlegung

## 28.2 Conditioning

## Conditioning

### Conditioning

**idea:** Apply backtracking with forward checking until the constraint graph **restricted to the remaining unassigned variables** decomposes or is a tree.

**remaining problem**  $\rightsquigarrow$  algorithms for simple constraint graphs

### cutset conditioning:

Choose variable order such that early variables form a small **cutset** (i.e., set of variables such that removing these variables results in an acyclic constraint graph).

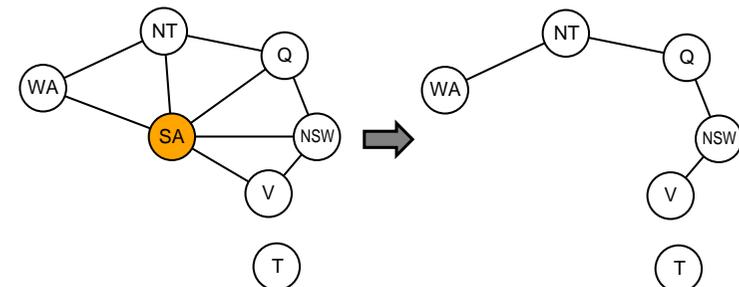
German: Cutset

**time complexity:**  $n$  variables,  $m < n$  in cutset,  
maximal domain size  $k$ :  $O(k^m \cdot (n - m)k^2)$

(Finding optimal cutsets is an NP-complete problem.)

## Conditioning: Example

**Australia example:** Cutset of size 1 suffices:



## 28.3 Tree Decomposition

## Tree Decomposition

basic idea of **tree decomposition**:

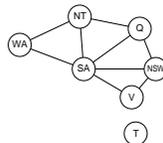
- ▶ Decompose constraint network into smaller **subproblems** (overlapping).
- ▶ Find solutions for the subproblems.
- ▶ Build overall solution based on the subsolutions.

more details:

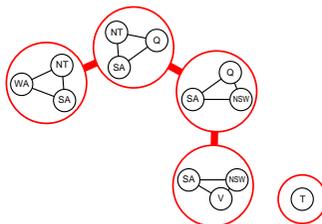
- ▶ “Overall solution building problem” based on subsolutions is a constraint network itself (**meta constraint network**).
- ▶ Choose subproblems in a way that the constraint graph of the meta constraint network is a **tree/forest**.  
 ~> build overall solution with efficient tree algorithm

## Tree Decomposition: Example

constraint network:



tree decomposition:



## Tree Decomposition: Definition

Definition (tree decomposition)

Consider a constraint network  $\mathcal{C}$  with variables  $V$ .

A **tree decomposition** of  $\mathcal{C}$  is a graph  $\mathcal{T}$  with the following properties.

**requirements on vertices:**

- ▶ Every **vertex** of  $\mathcal{T}$  corresponds to a subset of the variables  $V$ . Such a vertex (and corresponding variable set) is called a **subproblem** of  $\mathcal{C}$ .
- ▶ Every **variable** of  $V$  appears in **at least one** subproblem of  $\mathcal{T}$ .
- ▶ For every **nontrivial constraint**  $R_{uv}$  of  $\mathcal{C}$ , the variables  $u$  and  $v$  appear together in **at least one** subproblem in  $\mathcal{T}$ .

...

## Tree Decomposition: Definition

### Definition (tree decomposition)

Consider a constraint network  $\mathcal{C}$  with variables  $V$ .

A **tree decomposition** of  $\mathcal{C}$

is a graph  $\mathcal{T}$  with the following properties.

...

#### requirements on edges:

- ▶ For each variable  $v \in V$ , let  $\mathcal{T}_v$  be the set of vertices corresponding to the subproblems that contain  $v$ .
- ▶ For each variable  $v$ , the set  $\mathcal{T}_v$  is **connected**, i.e., each vertex in  $\mathcal{T}_v$  is reachable from every other vertex in  $\mathcal{T}_v$  without visiting vertices not contained in  $\mathcal{T}_v$ .
- ▶  $\mathcal{T}$  is **acyclic** (a tree/forest)

## Meta Constraint Network

meta constraint network  $\mathcal{C}^{\mathcal{T}} = \langle V^{\mathcal{T}}, \text{dom}^{\mathcal{T}}, (R_{uv}^{\mathcal{T}}) \rangle$

based on tree decomposition  $\mathcal{T}$

- ▶  $V^{\mathcal{T}} :=$  vertices of  $\mathcal{T}$  (i.e., subproblems of  $\mathcal{C}$  occurring in  $\mathcal{T}$ )
- ▶  $\text{dom}^{\mathcal{T}}(v) :=$  set of solutions of subproblem  $v$
- ▶  $R_{uv}^{\mathcal{T}} := \{ \langle s, t \rangle \mid s, t \text{ compatible solutions of subproblems } u, v \}$

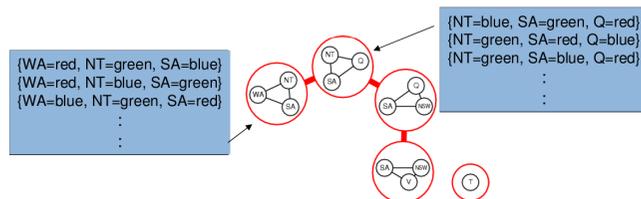
German: Meta-Constraintnetz

Solutions of two subproblems are called **compatible** if all overlapping variables are assigned identically.

## Solving with Tree Decompositions: Algorithm

### algorithm:

- ▶ Find **all solutions** for **all subproblems** in the decomposition and build a tree-like **meta constraint network**.
- ▶ Constraints in meta constraint network: subsolutions must be **compatible**.
- ▶ Solve meta constraint network with an algorithm for tree-like networks.



## Good Tree Decompositions

**goal:** each subproblem has as few variables as possible

- ▶ crucial: subproblem  $V'$  in  $\mathcal{T}$  with highest number of variables
- ▶ number of variables in  $V'$  minus 1 is called **width** of the decomposition
- ▶ best width over all decompositions: **tree width** of the constraint graph (computation is NP-complete)

**time complexity of solving algorithm based on tree decompositions:**

$O(nk^{w+1})$ , where  $w$  is width of decomposition

(requires specialized version of revise; otherwise  $O(nk^{2w+2})$ .)

## 28.4 Summary

## Summary: This Chapter

- ▶ Reduce **complex** constraint graphs to **simple** constraint graphs.
- ▶ **cutset conditioning**:
  - ▶ Choose **as few** variables as possible (cutset) such that an assignment to these variables yields a **remaining problem** which is structurally simple.
  - ▶ **search** over assignments of variables in cutset
- ▶ **tree decomposition**: build **tree-like** meta constraint network
  - ▶ meta variables: **groups** of original variables that jointly cover all variables and constraints
  - ▶ **values** correspond to consistent assignments to the groups
  - ▶ constraints between **overlapping** groups to ensure **compatibility**
  - ▶ overall algorithm exponential in **width** of decomposition (size of largest group)

## Summary: CSPs

### Constraint Satisfaction Problems (CSP)

**General** formalism for problems where

- ▶ values have to be assigned to variables
- ▶ such that the given constraints are satisfied.
- ▶ algorithms: **backtracking search + inference** (e.g., forward checking, arc consistency, path consistency)
- ▶ variable and value orders important
- ▶ more efficient: exploit **structure of constraint graph** (connected components; trees)

## More Advanced Topics

more advanced topics (not considered in this course):

- ▶ **backjumping**: backtracking over several layers
- ▶ **no-good learning**: infer additional constraints based on information collected during backtracking
- ▶ **local search methods** in the space of total, but not necessarily consistent assignments
- ▶ **tractable constraint classes**: identification of constraint types that allow for polynomial algorithms
- ▶ solutions of different quality: **constraint optimization problems (COP)**

↔ more than enough content for a one-semester course