

Foundations of Artificial Intelligence

25. Constraint Satisfaction Problems: Arc Consistency

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Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems:

- 22.–23. Introduction
- 24.–26. Basic Algorithms
 - 24. Backtracking
 - 25. Arc Consistency
 - 26. Path Consistency
- 27.–28. Problem Structure

Inference

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Derive additional constraints ([here](#): unary or binary) that are implied by the given constraints, i.e., that are satisfied in all solutions.

example: constraint network with variables v_1, v_2, v_3 with domain $\{1, 2, 3\}$ and constraints $v_1 < v_2$ and $v_2 < v_3$.

it follows:

- v_2 cannot be equal to 3
(new **unary constraint** = **tighter domain** of v_2)
- $R_{v_1 v_2} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$ can be tightened to $\{\langle 1, 2 \rangle\}$
(**tighter** binary constraint)
- $v_1 < v_3$
(“new” **binary constraint** = trivial constraint **tightened**)

Trade-Off Search vs. Inference

Inference formally

For a given constraint network $\mathcal{C}$, replace $\mathcal{C}$ with an **equivalent**, but **tighter** constraint network.

Trade-off:

- the **more complex** the inference, and
- the **more often** inference is applied,
- the **smaller** the resulting state space, but
- the **higher** the complexity **per search node**.

When to Apply Inference?

different possibilities to apply inference:

- once as **preprocessing** before search
 - **combined with search**: before recursive calls during backtracking procedure
 - already assigned variable $v \mapsto d$ corresponds to $\text{dom}(v) = \{d\}$
 $\rightsquigarrow$ more inferences possible
 - during backtracking, derived constraints have to be **retracted** because they were based on the given assignment
- $\rightsquigarrow$ powerful, but possibly expensive

Backtracking with Inference

```
function BacktrackingWithInference( $\mathcal{C}, \alpha$ ):
```

```
if  $\alpha$  is inconsistent with  $\mathcal{C}$ :  
    return inconsistent
```

```
if  $\alpha$  is a total assignment:  
    return  $\alpha$ 
```

```
 $\mathcal{C}' := \langle V, \text{dom}', (R'_{uv}) \rangle := \text{copy of } \mathcal{C}$   
apply inference to  $\mathcal{C}'$ 
```

```
if  $\text{dom}'(v) \neq \emptyset$  for all variables  $v$ :
```

```
    select some variable  $v$  for which  $\alpha$  is not defined
```

```
    for each  $d \in \text{copy of } \text{dom}'(v)$  in some order:
```

```
         $\alpha' := \alpha \cup \{v \mapsto d\}$ 
```

```
         $\text{dom}'(v) := \{d\}$ 
```

```
         $\alpha'' := \text{BacktrackingWithInference}(\mathcal{C}', \alpha')$ 
```

```
        if  $\alpha'' \neq \text{inconsistent}$ :
```

```
            return  $\alpha''$ 
```

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return inconsistent
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Backtracking with Inference: Discussion

- **Inference** is a placeholder:
different inference methods can be applied.
- Inference methods can recognize unsolvability (given α)
and indicate this by clearing the domain of a variable.
- Efficient implementations of inference are often **incremental**:
the previously assigned variable/value pair $v \mapsto d$ is taken
into account to speed up the inference computation.

Forward Checking

Forward Checking

We start with a simple inference method:

Forward Checking

Let α be a partial assignment.

Inference: For all unassigned variables v in α , remove all values from the domain of v that are in conflict with already assigned variable/value pairs in α .

$\rightsquigarrow$ definition of **conflict** as in the previous chapter

Incremental computation:

- When adding $v \mapsto d$ to the assignment, delete all pairs that conflict with $v \mapsto d$.

Forward Checking: Discussion

properties of forward checking:

- correct inference method (retains equivalence)
- affects domains (= unary constraints),
but not binary constraints
- consistency check at the beginning of the backtracking
procedure no longer needed (Why?)
- cheap, but often still useful inference method

~> apply at least forward checking in the backtracking procedure

In the following, we will consider more powerful inference methods.

Arc Consistency

Arc Consistency: Definition

Definition (Arc Consistent)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.

- (a) The variable $v \in V$ is **arc consistent** with respect to another variable $v' \in V$, if for every value $d \in \text{dom}(v)$ there exists a value $d' \in \text{dom}(v')$ with $\langle d, d' \rangle \in R_{vv'}$.
- (b) The constraint network $\mathcal{C}$ is **arc consistent**, if every variable $v \in V$ is arc consistent with respect to every other variable $v' \in V$.

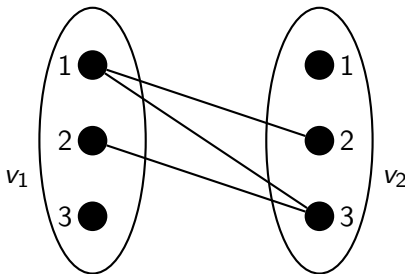
German: kantenkonsistent

remarks:

- definition for variable pair is not symmetrical
- v always arc consistent with respect to v' if the constraint between v and v' is trivial

Arc Consistency: Example

Consider a constraint network with variables v_1 and v_2 , domains $\text{dom}(v_1) = \text{dom}(v_2) = \{1, 2, 3\}$ and the constraint expressed by $v_1 < v_2$.



Arc consistency of v_1 with respect to v_2 and of v_2 with respect to v_1 are violated.

Enforcing Arc Consistency

- Enforcing arc consistency, i.e., removing values from $\text{dom}(v)$ that violate the arc consistency of v with respect to v' , is a correct inference method. (Why?)
- more powerful than forward checking (Why?)

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- Enforcing arc consistency, i.e., removing values from $\text{dom}(v)$ that violate the arc consistency of v with respect to v' , is a correct inference method. (Why?)
- more powerful than forward checking (Why?)
 - ↪ Forward checking is a special case:
enforcing arc consistency of all variables
with respect to the just assigned variable
corresponds to forward checking.

We will next consider algorithms that enforce arc consistency.

Processing Variable Pairs: revise

function revise($\mathcal{C}, v, v'$):

$\langle V, \text{dom}, (R_{uv}) \rangle := \mathcal{C}$

for each $d \in \text{dom}(v)$:

if there is no $d' \in \text{dom}(v')$ with $\langle d, d' \rangle \in R_{vv'}$:

remove d from $\text{dom}(v)$

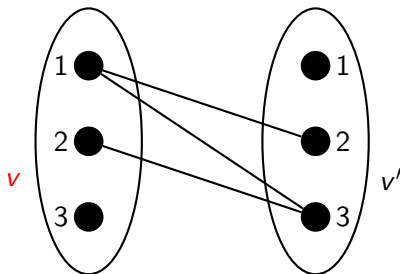
input: constraint network $\mathcal{C}$ and two variables v, v' of $\mathcal{C}$

effect: v arc consistent with respect to v' .

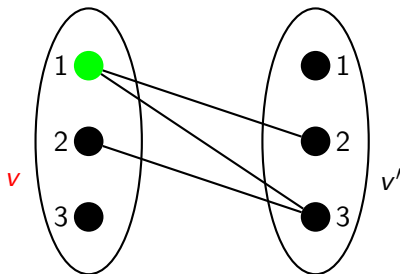
All violating values in $\text{dom}(v)$ are removed.

time complexity: $O(k^2)$, where k is maximal domain size

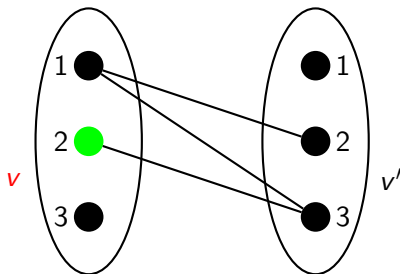
Example: revise



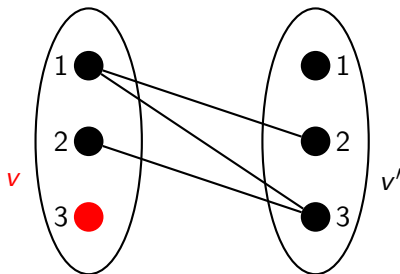
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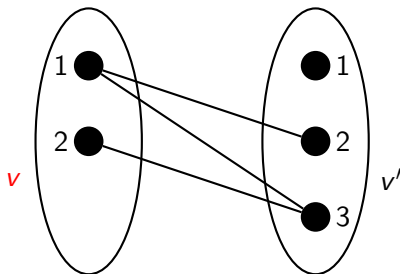
Example: revise



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Example: revise



Enforcing Arc Consistency: AC-1

```
function AC-1( $\mathcal{C}$ ):
```

```
   $\langle V, \text{dom}, (R_{uv}) \rangle := \mathcal{C}$ 
```

```
  repeat
```

```
    for each nontrivial constraint  $R_{uv}$ :
```

```
      revise( $\mathcal{C}, u, v$ )
```

```
      revise( $\mathcal{C}, v, u$ )
```

```
  until no domain has changed in this iteration
```

input: constraint network $\mathcal{C}$

effect: transforms $\mathcal{C}$ into equivalent arc consistent network

time complexity: ?

Enforcing Arc Consistency: AC-1

function AC-1($\mathcal{C}$):

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repeat

for each nontrivial constraint R_{uv} :

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until no domain has changed in this iteration

input: constraint network $\mathcal{C}$

effect: transforms $\mathcal{C}$ into equivalent arc consistent network

time complexity: $O(n \cdot e \cdot k^3)$, with n variables,
 e nontrivial constraints and maximal domain size k

AC-1: Discussion

- AC-1 does the job, but is rather inefficient.
- Drawback: Variable pairs are often checked again and again although their domains have remained unchanged.
- These (redundant) checks can be saved.

~> more efficient algorithm: AC-3

Enforcing Arc Consistency: AC-3

idea: store **potentially inconsistent** variable pairs in a queue

function AC-3($\mathcal{C}$):

$\langle V, \text{dom}, (R_{uv}) \rangle := \mathcal{C}$

$queue := \emptyset$

for each nontrivial constraint R_{uv} :

 insert $\langle u, v \rangle$ into $queue$

 insert $\langle v, u \rangle$ into $queue$

while $queue \neq \emptyset$:

 remove an arbitrary element $\langle u, v \rangle$ from $queue$

 revise($\mathcal{C}, u, v$)

if $\text{dom}(u)$ changed in the call to revise:

for each $w \in V \setminus \{u, v\}$ where R_{wu} is nontrivial:

 insert $\langle w, u \rangle$ into $queue$

AC-3: Discussion

- *queue* can be an arbitrary data structure that supports insert and remove operations (the order of removal does not affect the result)
- ↪ use data structure with fast insertion and removal, e.g., stack
- AC-3 has the same effect as AC-1:
it enforces arc consistency
- **proof idea:** invariant of the **while** loop:
If $\langle u, v \rangle \notin \text{queue}$, then u is arc consistent with respect to v

AC-3: Time Complexity

Proposition (time complexity of AC-3)

Let $\mathcal{C}$ be a constraint network with e nontrivial constraints and maximal domain size k .

The time complexity of AC-3 is $O(e \cdot k^3)$.

AC-3: Time Complexity (Proof)

Proof.

Consider a pair $\langle u, v \rangle$ such that there exists a nontrivial constraint R_{uv} or R_{vu} . (There are at most $2e$ of such pairs.)

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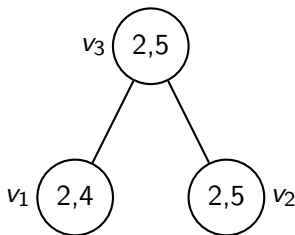
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Hence every pair $\langle u, v \rangle$ is inserted into the queue at most $k + 1$ times $\rightsquigarrow$ at most $O(ek)$ insert operations in total.

This bounds the number of **while** iterations by $O(ek)$, giving an overall time complexity of $O(ek) \cdot O(k^2) = O(ek^3)$. □

AC-3: Example

Consider a constraint network with three variables v_1 , v_2 , v_3 with $\text{dom}(v_1) = \{2, 4\}$ and $\text{dom}(v_2) = \text{dom}(v_3) = \{2, 5\}$ and the constraints $v_3|v_1$ and $v_3|v_2$ (“divides exactly”).



Queue

$\langle v_1, v_3 \rangle$

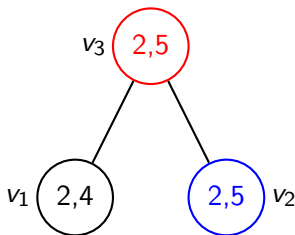
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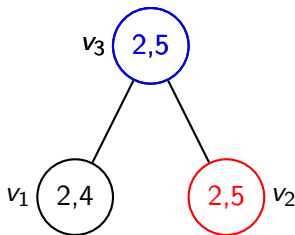
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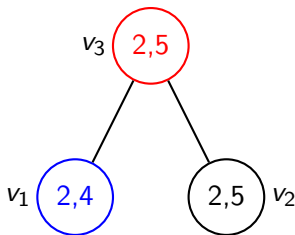
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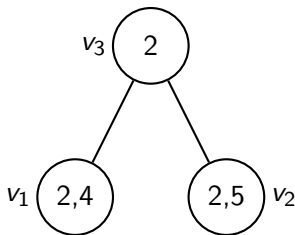


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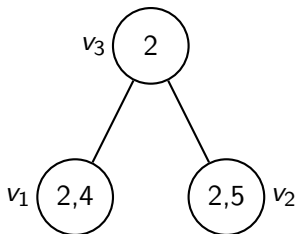


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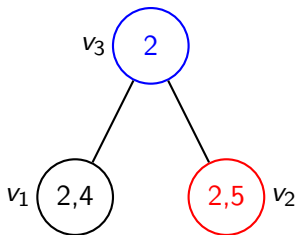
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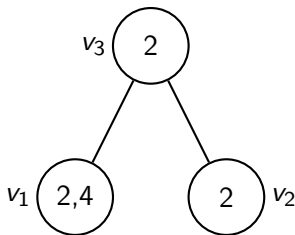
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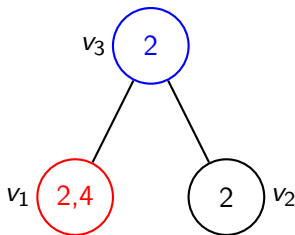


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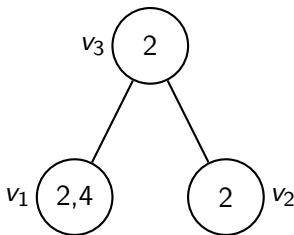


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Summary

Summary: Inference

- **inference**: derivation of additional constraints that are implied by the known constraints
- ⇒ **tighter equivalent** constraint network
- **trade-off** search vs. inference
- inference as **preprocessing** or **integrated** into backtracking

Summary: Forward Checking, Arc Consistency

- cheap and easy inference: **forward checking**
 - remove values that conflict with already assigned values
- more expensive and more powerful: **arc consistency**
 - iteratively remove values without a suitable “partner value” for another variable until fixed-point reached
 - efficient implementation of AC-3: $O(ek^3)$
with e : #nontrivial constraints, k : size of domain