# Theory of Computer Science

## E1. Complexity Theory: Motivation and Introduction

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# Overview: Course

contents of this course:

- logic ✓
  - ▷ How can knowledge be represented?
    How can reasoning be automated?
- automata theory and formal languages ✓
  - ▷ What is a computation?
- computability theory ✓
  - ▷ What can be computed at all?
- complexity theory
  - ▷ What can be computed efficiently?

# Overview: Complexity Theory

## Complexity Theory

E1. Motivation and Introduction

E2. P, NP and Polynomial Reductions

E3. Cook-Levin Theorem

E4. Some NP-Complete Problems, Part I

E5. Some NP-Complete Problems, Part II

# Further Reading (German)

## Literature for this Chapter (German)

Theoretische Informatik – kurz gefasst
by Uwe Schöning (5th edition)

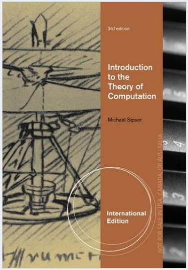
- Chapter 3.1

# Further Reading (English)

## Literature for this Chapter (English)

Introduction to the Theory of Computation
by Michael Sipser (3rd edition)

- Chapter 7.1



# Motivation

# A Scenario (1)

## Example Scenario

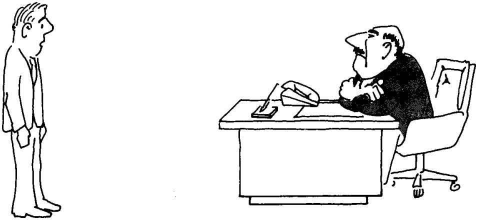
- You are a programmer at a logistics company.
- Your boss gives you the task of developing a program to optimize the route of a delivery truck:
  - The truck begins its route at the company depot.
  - It has to visit 50 stops.
  - You know the distances between all relevant locations (stops and depot).
  - Your program should compute a tour visiting all stops and returning to the depot on a shortest route.

# A Scenario (2)

## Example Scenario (ctd.)

- You work on the problem for weeks, but you do not manage to complete the task.
- All of your attempted programs
  - compute routes that are possibly suboptimal, or
  - do not terminate in reasonable time (say: within a month).
- What do you say to your boss?

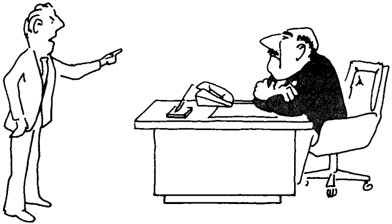
# What You Don't Want to Say



"I can't find an efficient algorithm,
I guess I'm just too dumb."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

# What You Would Like to Say



"I can't find an efficient algorithm,
because no such algorithm is possible!"

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 2

# What Complexity Theory Allows You to Say

"I can't find an efficient algorithm,
but neither can all these famous people."

Source: M. Garey & D. Johnson, Computers and Intractability, Freeman 1979, p. 3

# Why Complexity Theory?

**Complexity Theory**

Complexity theory tells us which problems can be solved quickly (“simple problems”) and which ones cannot (“hard problems”).

German: Komplexitätstheorie

- This is useful in practice because simple and hard problems require different techniques to solve.
- If we can show that a problem is hard we do not need to waste our time with the (futile) search for a “simple” algorithm.

# Why Reductions?

## Reductions

An important part of complexity theory are (polynomial) reductions that show how a given problem $P$ can be reduced to another problem $Q$.

German: Reduktionen

- useful for theoretical analysis of $P$ and $Q$ because it allows us to transfer our knowledge between them
- often also useful for practical algorithms for $P$: reduce $P$ to $Q$ and then use the best known algorithm for $Q$

# Test Your Intuition! (1)

- The following slide lists some graph problems.
- The input is always a directed graph $G = \langle V, E \rangle$.
- How difficult are the problems in your opinion?
- Sort the problems
  from easiest (= requires least amount of time to solve)
  to hardest (= requires most time to solve)
- no justification necessary, just follow your intuition!
- anonymous and not graded

# Test Your Intuition! (2)

1. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with minimal length.
2. Find a simple path (= without cycle) from $u \in V$ to $v \in V$ with maximal length.
3. Determine whether $G$ is strongly connected (every node is reachable from every other node).
4. Find a cycle (non-empty path from $u$ to $u$ for any $u \in V$; multiple visits of nodes are allowed).
5. Find a cycle that visits all nodes.
6. Find a cycle that visits a given node $u$.
7. Find a path that visits all nodes without repeating a node.
8. Find a path that uses all edges without repeating an edge.

# How to Measure Runtime?

# How to Measure Runtime?

- Time complexity is a way to measure how much time it takes to solve a problem.
- How can we define such a measure appropriately?

German: Zeitkomplexität/Zeitaufwand

# Example Statements about Runtime

Example statements about runtime:

- "Running `sort /usr/share/dict/words` on the computer `dakar` takes 0.105 seconds."
- "With a 1 MiB input file, `sort` takes at most 1 second on a modern computer."
- "Quicksort is faster than sorting by insertion."
- "Sorting by insertion is slow."

$\rightsquigarrow$ Very different statements with different pros and cons.

# Precise Statements vs. General Statements

**Example Statement about Runtime**

"Running `sort /usr/share/dict/words`
on the computer `dakar` takes 0.105 seconds."

advantage: very precise

disadvantage: not general

- input-specific:
  What if we want to sort other files?
- machine-specific:
  What happens on a different computer?
- even situation-specific:
  Will we get the same result tomorrow that we got today?

# General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

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## 1. General Inputs

Instead of concrete inputs, we talk about general types of input:

- Example: runtime to sort an input of size $n$ in the worst case
- Example: runtime to sort an input of size $n$ in the average case

here: runtime for input size $n$ in the worst case

# General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

## 2. Ignoring Details

Instead of exact formulas for the runtime we specify the order of magnitude:

- Example: instead of saying that we need time $\lceil 1.2n \log n \rceil - 4n + 100$, we say that we need time $O(n \log n)$.
- Example: instead of saying that we need time $O(n \log n)$, $O(n^2)$ or $O(n^4)$, we say that we need polynomial time.

here: What can be computed in polynomial time?

# General Statements about Runtime

In this course we want to make general statements about runtime. We accomplish this in three ways:

## 3. Abstract Cost Measures

Instead of the runtime on a concrete computer we consider a more abstract cost measure:

- Example: count the number of executed machine code statements
- Example: count the number of executed Java byte code statements
- Example: count the number of element comparisons of a sorting algorithms

here: count the computation steps of a Turing machine (polynomially equivalent to other measures)

# Questions

Questions?

# Decision Problems

# Decision Problems

- As before, we simplify our investigation by restricting our attention to decision problems.
- More complex computational problems can be solved with multiple queries for an appropriately defined decision problem ("playing 20 questions").
- Formally, decision problems are languages (as before), but we use an informal "given" / "question" notation where possible.

# Example: Decision vs. General Problem (1)

## Definition (Hamilton Cycle)

Let $G = \langle V, E \rangle$ be a (directed or undirected) graph.

A Hamilton cycle of $G$ is a sequence of vertices in $V$, $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- $\pi$ is a path: there is an edge from $v_i$ to $v_{i+1}$ for all $0 \leq i < n$
- $\pi$ is a cycle: $v_0 = v_n$
- $\pi$ is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- $\pi$ is Hamiltonian: all nodes of $V$ are included in $\pi$

German: Hamiltonkreis/Hamiltonzyklus

# Example: Decision vs. General Problem (2)

## Example (Hamilton Cycles in Directed Graphs)

$\mathcal{P}$: general problem DIRHAMILTONCYCLEGEN

- Input: directed graph $G = \langle V, E \rangle$
- Output: a Hamilton cycle of $G$ or a message that none exists

$\mathcal{E}$: decision problem DIRHAMILTONCYCLE

- Given: directed graph $G = \langle V, E \rangle$
- Question: Does $G$ contain a Hamilton cycle?

These problems are polynomially equivalent:
from a polynomial algorithm for one of the problems
one can construct a polynomial algorithm for the other problem.
(Without proof.)

# Algorithms for Decision Problems

## Algorithms for decision problems:

- Where possible, we specify algorithms for decision problems in pseudo-code (similar to WHILE programs).
- Since they are only yes/no questions, we do not have to return a general result.
- Instead we use the statements
  - **ACCEPT** to accept the given input ("yes" answer) and
  - **REJECT** to reject it ("no" answer).
- Where we must be more formal, we use Turing machines and the notion of accepting from chapter C7.

# Questions

Questions?

# Nondeterminism

# Nondeterminism

- To develop complexity theory, we need the algorithmic concept of nondeterminism.
- already known for Turing machines ($\rightsquigarrow$ chapter C7):
  - An NTM can have more than one possible successor configuration for a given configuration.
  - Input $x$ is accepted if there is at least one possible computation (configuration sequence) that leads to an end state.
- Here we analogously introduce nondeterminism for pseudo-code (or WHILE programs).

German: Nichtdeterminismus

# Nondeterministic Algorithms

nondeterministic algorithms:

- All constructs of deterministic algorithms are also allowed in nondeterministic algorithms: **IF**, **WHILE**, etc.
- Additionally, there is a nondeterministic assignment:

$$\textbf{GUESS } x_i \in \{0, 1\}$$

where $x_i$ is a program variable.

German: nichtdeterministische Zuweisung

# Nondeterministic Algorithms: Acceptance

- Meaning of **GUESS** $x_i \in \{0, 1\}$:
  $x_i$ is assigned either the value 0 or the value 1.
- This implies that the behavior of the program on a given input is no longer uniquely defined: there are multiple possible execution paths.
- The program accepts a given input if at least one execution path leads to an **ACCEPT** statement.
- Otherwise, the input is rejected.

Note: asymmetry between accepting and rejecting!
(cf. semi-decidability)

# More Complex GUESS Statements

- We will also guess more than one bit at a time:

  $$\textbf{GUESS } x \in \{1, 2, \ldots, n\}$$

  or more generally

  $$\textbf{GUESS } x \in S$$

  for a set $S$.
- These are abbreviations and can be split into $\lceil \log_2 n \rceil$ (or $\lceil \log_2 |S| \rceil$) "atomic" **GUESS** statements.

# Example: Nondeterministic Algorithms (1)

**Example (DirHamiltonCycle)**

input: directed graph $G = \langle V, E \rangle$

*start* := an arbitrary node from $V$
*current* := *start*
*remaining* := $V \setminus \{start\}$
**WHILE** $remaining \neq \emptyset$:
    **GUESS** $next \in remaining$
    **IF** $\langle current, next \rangle \notin E$:
        **REJECT**
    $remaining := remaining \setminus \{next\}$
    $current := next$
**IF** $\langle current, start \rangle \in E$:
    **ACCEPT**

# Example: Nondeterministic Algorithms (2)

- With appropriate data structures, this algorithm solves the problem in $O(n \log n)$ program steps, where $n = |V| + |E|$ is the size of the input.
- How many steps would a deterministic algorithm need?

# Guess and Check

- The DirHamiltonCycle example illustrates a general design principle for nondeterministic algorithms:

  guess and check

- In general, nondeterministic algorithms can solve a problem by first guessing a “solution” and then verifying that it is indeed a solution. (In the example, these two steps are interleaved.)
- If solutions to a problem can be efficiently verified, then the problem can also be efficiently solved if nondeterminism may be used.

German: Raten und Prüfen

# The Power of Nondeterminism

- Nondeterministic algorithms are very powerful because they can "guess" the "correct" computation step.
- Or, interpreted differently: they go through many possible computations "in parallel", and it suffices if one of them is successful.
- Can they solve problems efficiently (in polynomial time) which deterministic algorithms cannot solve efficiently?
- This is the big question!

# Questions

Questions?

# Summary

# Summary (1)

- Complexity theory deals with the question which problems can be solved efficiently and which ones cannot.
- here: focus on what can be computed in polynomial time
- To formalize this, we use Turing machines, but other formalisms are polynomially equivalent.
- We consider decision problems, but the results directly transfer to general computational problems.

# Summary (2)

important concept: nondeterminism

- Nondeterministic algorithms can "guess", i. e., perform multiple computations "at the same time".
- An input receives a "yes" answer if at least one computation path accepts it.
- in NTMs: with nondeterministic transitions ($\delta(q, a)$ contains multiple elements)
- in pseudo-code: with **GUESS** statements