Theory of Computer Science

D8. Rice's Theorem and Other Undecidable Problems

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Overview: Computability Theory

Computability Theory

- imperative models of computation:
  - D1. Turing-Computability
  - D2. LOOP- and WHILE-Computability
  - D3. GOTO-Computability
- functional models of computation:
  - D4. Primitive Recursion and μ-Recursion
  - D5. Primitive/μ-Recursion vs. LOOP-/WHILE-Computability
- undecidable problems:
  - D6. Decidability and Semi-Decidability
  - D7. Halting Problem and Reductions
  - D8. Rice's Theorem and Other Undecidable Problems
    ~~Post's Correspondence Problem~~
    ~~Undecidable Grammar Problems~~
    ~~Gödel's Theorem and Diophantine Equations~~

Further Reading (German)

Literature for this Chapter (German)

Theoretische Informatik – kurz gefasst
by Uwe Schöning (5th edition)

- Chapter 2.6
- Outlook: Chapters 2.7–2.9

Further Reading (English)

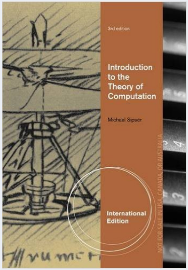
Literature for this Chapter (English)

Introduction to the Theory of Computation
by Michael Sipser (3rd edition)

- Chapters 5.1 and 5.3
- Outlook: Chapters 5.2 and 6.2

Notes:

- Sipser's definitions differ from ours.



Other Halting Problem Variants

General Halting Problem (1)

Definition (General Halting Problem)

The general halting problem or halting problem is the language

$$H = \{w\#x \in \{0,1,\#\}^* \mid w, x \in \{0,1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

German: allgemeines Halteproblem, Halteproblem

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Theorem (Undecidability of General Halting Problem)

The general halting problem is undecidable.

Intuition: if the special case K is not decidable,
then the more general problem H definitely cannot be decidable.

General Halting Problem (2)

Proof.

We show $K \leq H$ (Reminder: K is the special halting problem).

We define $f : \{0,1\}^* \to \{0,1,\#\}^*$ as $f(w) := w\#w$.

f is clearly total and computable, and

$$\begin{aligned} & w \in K \\ \text{iff } & M_w \text{ started on } w \text{ terminates} \\ \text{iff } & w\#w \in H \\ \text{iff } & f(w) \in H. \end{aligned}$$

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\end{aligned}
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Therefore f is a reduction from K to H.
Because K is undecidable, H is also undecidable. $\square$

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The halting problem on the empty tape is the language

$$H_0 = \{w \in \{0,1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

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Theorem (Undecidability of Halting Problem on Empty Tape)

The halting problem on the empty tape is undecidable.

Halting Problem on Empty Tape (2)

Proof.

We show $H \leq H_0$.

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that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

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- Test if z has the form $w\#x$ with $w, x \in \{0, 1\}^*$.
- If not, return any word that is not in H_0
  (e. g., encoding of a TM that instantly starts an endless loop).
- If yes, split z into w and x.

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- If yes, split z into w and x.
- Decode w to a TM M_2.

...

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
  - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
  - otherwise, stop immediately

Halting Problem on Empty Tape (3)

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- Construct a TM M_1 that behaves as follows:
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- Construct TM M that first runs M_1 and then M_2.

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f is total and (with some effort) computable. Also:

$$\begin{aligned} z \in H &\text{ iff } z = w\#x \text{ and } M_w \text{ run on } x \text{ terminates} \\ &\text{ iff } M_{f(z)} \text{ started on empty tape terminates} \\ &\text{ iff } f(z) \in H_0 \end{aligned}$$

$\rightsquigarrow H \leq H_0 \rightsquigarrow H_0$ undecidable $\square$

Questions

Questions?

Rice's Theorem

Rice's Theorem (1)

- We have shown that a number of (related) problems are undecidable:
  - special halting problem K
  - general halting problem H
  - halting problem on empty tape H_0
- Many more results of this type could be shown.
- Instead, we prove a much more general result, Rice's theorem, which shows that a very large class of different problems are undecidable.
- Rice's theorem can be summarized informally as: every question about what a given Turing machine computes is undecidable.

Rice's Theorem (2)

Theorem (Rice's Theorem)

Let $\mathcal{R}$ be the class of all computable functions.
Let $\mathcal{S}$ be an arbitrary subset of $\mathcal{R}$ except $\mathcal{S} = \emptyset$ or $\mathcal{S} = \mathcal{R}$.
Then the language

$$C(\mathcal{S}) = \{w \in \{0, 1\}^* \mid \text{the function computed by } M_w \text{ is in } \mathcal{S}\}$$

is undecidable.

German: Satz von Rice
Question: why the restriction to $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \neq \mathcal{R}$?

Extension (without proof): in most cases neither $C(\mathcal{S})$ nor $\overline{C(\mathcal{S})}$ is semi-decidable. (But there are sets $\mathcal{S}$ for which one of the two languages is semi-decidable.)

Rice's Theorem (3)

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Let Ω be the function that is undefined everywhere.

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Let $q \in \mathcal{R} \setminus \mathcal{S}$ be an arbitrary computable function outside of $\mathcal{S}$ (exists because $\mathcal{S} \subseteq \mathcal{R}$ and $\mathcal{S} \neq \mathcal{R}$).

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Let Q be a Turing machine that computes q. ...

Rice's Theorem (4)

Proof (continued).

Consider function $f : \{0,1\}^* \to \{0,1\}^*$,
where $f(w)$ is defined as follows:

- Construct TM M that first behaves on input y like M_w on the empty tape (independently of what y is).
- Afterwards (if that computation terminates!) M clears the tape, creates the start configuration of Q for input y and then simulates Q.
- $f(w)$ is the encoding of this TM M

Rice's Theorem (4)

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f is total and computable. ...

Rice's Theorem (5)

Proof (continued).

Which function is computed by the TM encoded by $f(w)$?

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For all words $w \in \{0,1\}^*$:

$$\begin{aligned} w \in H_0 &\Longrightarrow M_w \text{ terminates on } \varepsilon \\ &\Longrightarrow M_{f(w)} \text{ computes the function } q \\ &\Longrightarrow \text{the function computed by } M_{f(w)} \text{ is not in } \mathcal{S} \\ &\Longrightarrow f(w) \notin C(\mathcal{S}) \end{aligned}$$

...

Rice's Theorem (6)

Proof (continued).

Further:

$$\begin{aligned} w \notin H_0 &\implies M_w \text{ does not terminate on } \varepsilon \\ &\implies M_{f(w)} \text{ computes the function } \Omega \\ &\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S} \\ &\implies f(w) \in C(\mathcal{S}) \end{aligned}$$

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Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

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Therefore, f is a reduction of $\bar{H}_0$ to $C(\mathcal{S})$.

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Since H_0 is undecidable, $\bar{H}_0$ is also undecidable.

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We can conclude that $C(\mathcal{S})$ is undecidable. ...

Rice's Theorem (7)

Proof (continued).

Case 2: $\Omega \notin \mathcal{S}$

Analogous to Case 1 but this time choose $q \in \mathcal{S}$.

The corresponding function f then reduces H_0 to $C(\mathcal{S})$.

Thus, it also follows in this case that $C(\mathcal{S})$ is undecidable. $\square$

Rice's Theorem: Consequences

Was it worth it?

We can now know conclude immediately that (for example) the following informally specified problems are all undecidable:

- Does a given TM compute a constant function?
- Does a given TM compute a total function (i. e. will it always terminate, and in particular terminate in a "correct" configuration)?
- Is the output of a given TM always longer than its input?
- Does a given TM compute the identity function?
- Does a given TM compute the computable function f?
- ...

Rice's Theorem: Practical Applications

Practical problems that are undecidable due to Rice's theorem:

- automated debugging:
  - Can a given variable ever receive a `null` value?
  - Can a given assertion in a program ever trigger?
  - Can a given buffer ever overflow?
- virus scanners and other software security analysis:
  - Can this code do something harmful?
  - Is this program vulnerable to SQL injections?
  - Can this program lead to a privilege escalation?
- optimizing compilers:
  - Is this dead code?
  - Is this a constant expression?
  - Can pointer aliasing happen here?
  - Is it safe to parallelize this code path?
- parallel program analysis:
  - Is a deadlock possible here?
  - Can a race condition happen here?

Questions

Questions?

Outlook: Further Undecidable Problems

And What Else?

- Here we conclude our discussion of undecidable problems.
- Many more undecidable problems exist.
- In this section, we briefly discuss some further classical results.

Post's Correspondence Problem

Post's Correspondence Problem (PCP)

Given: a set of "domino" types

A:	B:	C:
1	10	011
101	00	11

Question: Can we lay a (non-empty) sequence of dominoes such that the top and bottom word match?
(We may use each type as often as we want.)

1	011	10	011
101	11	00	11
A	C	B	C

$\rightsquigarrow$ undecidable by reduction from Halting problem
(see Schöning, Chapter 2.7)

German: Postsches Korrespondenzproblem

Undecidable Grammar Problems

Some Grammar Problems

Given context-free grammars G_1 and G_2, ...

- ... is $\mathcal{L}(G_1) \cap \mathcal{L}(G_2) = \emptyset$?
- ... is $|\mathcal{L}(G_1) \cap \mathcal{L}(G_2)| = \infty$?
- ... is $\mathcal{L}(G_1) \cap \mathcal{L}(G_2)$ context-free?
- ... is $\mathcal{L}(G_1) \subseteq \mathcal{L}(G_2)$?
- ... is $\mathcal{L}(G_1) = \mathcal{L}(G_2)$?

Given a context-sensitive grammar G, ...

- ... is $\mathcal{L}(G) = \emptyset$?
- ... is $|\mathcal{L}(G)| = \infty$?

$\rightsquigarrow$ all undecidable by reduction from PCP
(see Schöning, Chapter 2.8)

Gödel's First Incompleteness Theorem (1)

Definition (Arithmetic Formula)

An arithmetic formula is a closed predicate logic formula using

- constant symbols 0 and 1,
- function symbols $+$ and $\cdot$, and
- no relation symbols.

It is called true if it is true under the usual interpretation of 0, 1, $+$ and $\cdot$ over $\mathbb{N}_0$.

German: arithmetische Formel

Gödel's First Incompleteness Theorem (2)

Gödel's First Incompleteness Theorem

The problem of deciding if a given arithmetic formula is true is undecidable.

Moreover, neither it nor its complement are semi-decidable.

As a consequence, there exists no sound and complete proof system for arithmetic formulas.

$\rightsquigarrow$ proof idea: relate arithmetic formulas to halting problem for WHILE programs (see Schöning, Chapter 2.9)

German: erster Gödelscher Unvollständigkeitssatz

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Summary

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undecidable but semi-decidable problems:

- special halting problem a.k.a. self-application problem (from previous chapter)
- general halting problem
- halting problem on empty tape

Rice's theorem:

- "In general one cannot determine algorithmically what a given program (or Turing machine) computes."

further undecidable problems:

- Post's Correspondence Problem
- many grammar problems
- proving or refuting the truth of an arithmetic formula