

Theory of Computer Science

C7. Context-Sensitive and Type-0 Languages

Malte Helmert

University of Basel

April 13, 2016

Theory of Computer Science

April 13, 2016 — C7. Context-Sensitive and Type-0 Languages

C7.1 Context-Sensitive and General Grammars

C7.2 Turing Machines

C7.3 Turing Machines vs. Grammars

C7.4 Closure Properties and Decidability

C7.5 Summary

C7.1 Context-Sensitive and General Grammars

Repetition: (Context-Sensitive) Grammars

Definition (Grammar)

A **grammar** is a 4-tuple $\langle \Sigma, V, P, S \rangle$ with:

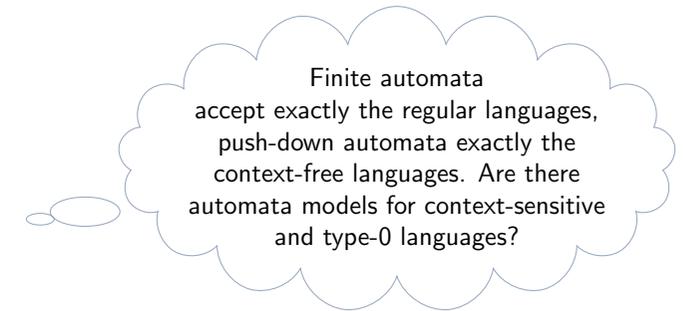
- ▶ Σ finite alphabet of terminal symbols
- ▶ V finite set of variables (with $V \cap \Sigma = \emptyset$)
- ▶ $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ finite set of rules
- ▶ $S \in V$ start variable

Type 0 and type 1:

- ▶ **Every grammar** is **type 0**.
- ▶ A grammar is **context-sensitive (type-1)** if all rules $w_1 \rightarrow w_2$ satisfy $|w_1| \leq |w_2|$.
 - ▶ **only exception:** $S \rightarrow \varepsilon$ is allowed for the start symbol S if S never occurs on a right-hand side.

C7.2 Turing Machines

Automata for Type-1 and Type-0 Languages?



Yes! \rightsquigarrow Turing machines
German: Turingmaschinen

Picture courtesy of imagerymajestic / FreeDigitalPhotos.net

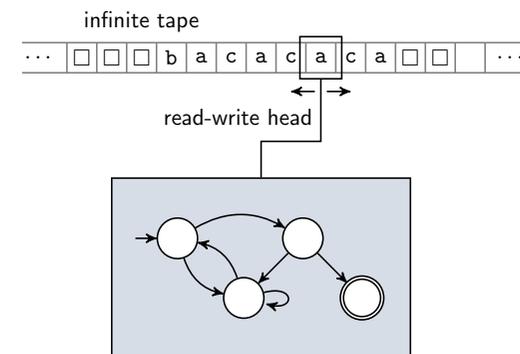
Alan Turing (1912–1954)



Picture courtesy of Jon Callas / wikimedia commons

- ▶ British logician, mathematician, cryptanalyst and computer scientist
- ▶ most important work (for us): *On Computable Numbers, with an Application to the Entscheidungsproblem*
 \rightsquigarrow Turing machines
- ▶ collaboration on **Enigma decryption**
- ▶ conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- ▶ **Turing award** most important science award in computer science

Turing Machines: Conceptually



Nondeterministic Turing Machine: Definition

Definition (Nondeterministic Turing Machine)

A nondeterministic **Turing machine (NTM)** is given by a 7-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ with:

- ▶ Q finite non-empty set of **states**
- ▶ $\Sigma \neq \emptyset$ finite **input alphabet**
- ▶ $\Gamma \supset \Sigma$ finite **tape alphabet**
- ▶ $\delta : (Q \setminus E) \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R, N\})$ **transition function**
- ▶ $q_0 \in Q$ **start state**
- ▶ $\square \in \Gamma \setminus \Sigma$ **blank symbol**
- ▶ $E \subseteq Q$ **end states**

German: Turingmaschine, Zustände, Eingabealphabet, Bandalphabet, Übergangsfunktion, Startzustand, Blank-Symbol, Endzustände

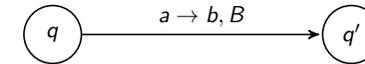
Turing Machine: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ be an NTM.

What is the Intuitive Meaning of the Transition Function δ ?

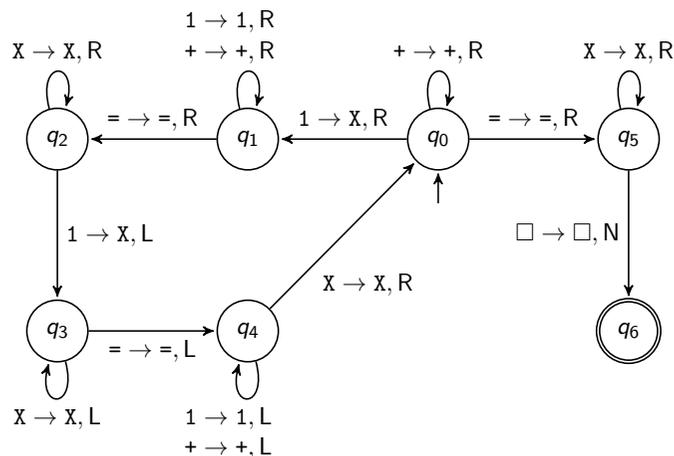
$\langle q', b, B \rangle \in \delta(q, a)$:

- ▶ If M is in state q and reads a , then
- ▶ M can transition to state q' in the next step,
- ▶ replacing a with b ,
- ▶ and moving the head in direction $B \in \{L, R, N\}$, where:
 - ▶ **L**: one step to the **left**,
 - ▶ **R**: one step to the **right**,
 - ▶ **N**: **neutral** (no) movement.



Nondeterministic Turing Machine: Example

$M = \langle \{q_0, q_1, \dots, q_6\}, \{1, +, =\}, \{1, +, =, X, \square\}, \delta, q_0, \square, \{q_6\} \rangle$



Turing Machine: Configuration

Definition (Configuration of a Turing Machine)

A **configuration** of a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ is given by a triple $c \in \Gamma^* \times Q \times \Gamma^+$.

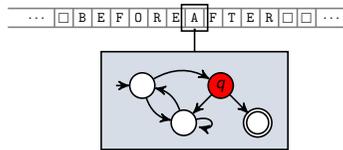
German: Konfiguration

Configuration $\langle w_1, q, w_2 \rangle$ intuitively means that

- ▶ the non-empty or already visited part of the tape contains the word $w_1 w_2$,
- ▶ the read-write head is on the first symbol of w_2 , and
- ▶ the TM is in state q .

Turing Machine Configurations: Example

Example



configuration
 $\langle \square \text{BEFORE}, q, \text{AFTER} \square \square \rangle$.

Turing Machine: Step

Definition (Transition/Step of a Turing Machine)

An NTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ can transition from configuration c to configuration c' in one step ($c \vdash_M c'$) according to the following rules:

- ▶ $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m, q', cb_2 \dots b_n \rangle$
if $\langle q', c, N \rangle \in \delta(q, b_1)$, $m \geq 0$, $n \geq 1$
- ▶ $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_{m-1}, q', a_m cb_2 \dots b_n \rangle$
if $\langle q', c, L \rangle \in \delta(q, b_1)$, $m \geq 1$, $n \geq 1$
- ▶ $\langle \varepsilon, q, b_1 \dots b_n \rangle \vdash_M \langle \varepsilon, q', \square cb_2 \dots b_n \rangle$
if $\langle q', c, L \rangle \in \delta(q, b_1)$, $n \geq 1$
- ▶ $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m c, q', b_2 \dots b_n \rangle$
if $\langle q', c, R \rangle \in \delta(q, b_1)$, $m \geq 0$, $n \geq 2$
- ▶ $\langle a_1 \dots a_m, q, b_1 \rangle \vdash_M \langle a_1 \dots a_m c, q', \square \rangle$
if $\langle q', c, R \rangle \in \delta(q, b_1)$, $m \geq 0$

Turing Machines: Reachability of Configurations

Definition (Reachable Configuration)

Configuration c' is **reachable** from configuration c in NTM M ($c \vdash_M^* c'$) if there are configurations c_0, \dots, c_n ($n \geq 0$) where

- ▶ $c_0 = c$,
- ▶ $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, n-1\}$, and
- ▶ $c_n = c'$.

German: c' ist in M von c erreichbar

Turing Machines: Recognized Words

Definition (Recognized Word of a Turing Machine)

NTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ **recognizes the word** $w = a_1 \dots a_n$ iff an **accepting configuration** (where M is in an end state) is reachable from the **start configuration**:

$$M \text{ recognizes } w \text{ iff } \langle \varepsilon, q_0, w \rangle \vdash_M^* \langle w_1, q, w_2 \rangle \\ \text{for some } q \in E, w_1 \in \Gamma^*, w_2 \in \Gamma^+$$

special case: for $w = \varepsilon$ the start configuration is $\langle \varepsilon, q_0, \square \rangle$ rather than $\langle \varepsilon, q_0, \varepsilon \rangle$

German: M erkennt w , akzeptierende Konfiguration, Startkonfiguration

example: blackboard

Turing Machines: Accepted Language

Definition (Accepted Language of an NTM)

Let M be an NTM with input alphabet Σ .

The **language accepted by M** is defined as

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ recognizes } w\}.$$

German: erkannte Sprache

example: blackboard

C7.3 Turing Machines vs. Grammars

One Automata Model for Two Grammar Types?

Don't we need
different automata models for
context-sensitive and type-0
languages?



Picture courtesy of stockimages / FreeDigitalPhotos.net

Linear Bounded Automata: Idea

- ▶ **Linear bounded automata** are NTMs that may only use the **part of the tape occupied by the input word**.
- ▶ one way of formalizing this: NTMs where blank symbol may never be replaced by a different symbol

Linear Bounded Turing Machines: Definition

Definition (Linear Bounded Automata)

An NTM $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$

is called a **linear bounded automaton (LBA)**

if for all $q \in Q \setminus E$ and all transition rules $\langle q', c, y \rangle \in \delta(q, \square)$ we have $c = \square$.

German: linear beschränkte Turingmaschine

LBAs Accept Type-1 Languages

Theorem

The languages that can be accepted by linear bounded automata are exactly the context-sensitive (type-1) languages.

Without proof.

proof sketch for grammar \Rightarrow NTM direction:

- ▶ computation of the NTM follows the production of the word in the grammar **in opposite order**
- ▶ accept when only start symbol (and blanks) are left on the tape
- ▶ because language is context-sensitive, we never need additional space on the tape (empty word needs special treatment)

NTMs Accept Type-0 Languages

Theorem

The languages that can be accepted by nondeterministic Turing machines are exactly the type-0 languages.

Without proof.

proof sketch for grammar \Rightarrow NTM direction:

- ▶ analogous to previous proof
- ▶ for grammar rules $w_1 \rightarrow w_2$ with $|w_1| > |w_2|$, we must "insert" symbols into the existing tape content; this is a bit tedious, but not very difficult

Deterministic Turing Machines

Definition (Deterministic Turing Machine)

A **deterministic Turing machine (DTM)** is a Turing machine

$M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$ with

$\delta : (Q \setminus E) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, N\}$.

German: deterministische Turingmaschine

Deterministic Turing Machines vs. Type-0 Languages

Theorem

For every type-0 language L there is a deterministic Turing machine M with $\mathcal{L}(M) = L$.

Without proof.

proof sketch:

- ▶ Let M' be an NTM with $\mathcal{L}(M') = L$.
- ▶ It is possible to construct a DTM that systematically searches for an accepting configuration in the computation tree of M' .

Note: It is an open problem whether an analogous theorem holds for type-1 languages and deterministic LBAs.

C7.4 Closure Properties and Decidability

Closure Properties

	Intersection	Union	Complement	Product	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽¹⁾
Type 0	Yes ⁽²⁾	Yes ⁽¹⁾	No ⁽³⁾	Yes ⁽¹⁾	Yes ⁽¹⁾

Proofs?

- (1) proof via grammars, similar to context-free cases
- (2) without proof
- (3) proof in later chapters (part D)

Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes ⁽¹⁾	No ⁽³⁾	No ⁽²⁾	No ⁽²⁾
Type 0	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾

Proofs?

- (1) same argument we used for context-free languages
- (2) because already undecidable for context-free languages
- (3) without proof
- (4) proofs in later chapters (part D)

C7.5 Summary

Summary

- ▶ Turing machines only have finitely many states but an **unbounded tape** as “memory”.
- ▶ **Turing machines** accept exactly the **type-0** languages. This is also true for **deterministic Turing machines**.
- ▶ **Linear bounded automata** accept exactly the **context-sensitive** languages.
- ▶ The context-sensitive and type-0 languages are **closed** under **almost all** usual operations.
 - ▶ exception: **type-0 not closed** under **complement**
- ▶ For context-sensitive and type-0 languages **almost no problem is decidable**.
 - ▶ exception: **word problem** for **context-sensitive lang.** decidable