

Theory of Computer Science

C1. Formal Languages and Grammars

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Introduction

Example: Propositional Formulas

from the logic part:

Definition (Syntax of Propositional Logic)

Let A be a set of **atomic propositions**. The set of **propositional formulas** (over A) is inductively defined as follows:

- Every **atom** $a \in A$ is a propositional formula over A .
- If φ is a propositional formula over A , then so is its **negation** $\neg\varphi$.
- If φ and ψ are propositional formulas over A , then so is the **conjunction** $(\varphi \wedge \psi)$.
- If φ and ψ are propositional formulas over A , then so is the **disjunction** $(\varphi \vee \psi)$.

Example: Propositional Formulas

Let S_A be the set of all propositional formulas over A .

Such sets of symbol sequences (or **words**) are called **languages**.

Sought: General concepts to define such (often infinite) languages with finite descriptions.

- today: **grammars**
- later: automata

Example: Propositional Formulas

Example (Grammar for $S_{\{a,b,c\}}$)

Grammar variables $\{F, A, N, C, D\}$ with start variable F , terminal symbols $\{a, b, c, \neg, \wedge, \vee, (,)\}$ and rules

$$\begin{array}{lll} F \rightarrow A & A \rightarrow a & N \rightarrow \neg F \\ F \rightarrow N & A \rightarrow b & C \rightarrow (F \wedge F) \\ F \rightarrow C & A \rightarrow c & D \rightarrow (F \vee F) \\ F \rightarrow D & & \end{array}$$

Start with F . In each step, replace a left-hand side of a rule with its right-hand side until no more variables are left:

$$\begin{aligned} F &\Rightarrow N \Rightarrow \neg F \Rightarrow \neg D \Rightarrow \neg(F \vee F) \Rightarrow \neg(A \vee F) \Rightarrow \neg(b \vee F) \\ &\Rightarrow \neg(b \vee A) \Rightarrow \neg(b \vee c) \end{aligned}$$

Alphabets and Formal Languages

Alphabets and Formal Languages

Definition (Alphabets, Words and Formal Languages)

An **alphabet** Σ is a finite non-empty set of **symbols**.

German: Alphabet, Zeichen/Symbole, leeres Wort, formale Sprache

Example

$$\Sigma = \{a, b\}$$

Alphabets and Formal Languages

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An **alphabet** Σ is a finite non-empty set of **symbols**.

A **word over Σ** is a finite sequence of elements from Σ .

The **empty word** (the empty sequence of elements) is denoted by ε .

Σ^* denotes the set of all words over Σ .

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Example

$$\Sigma = \{a, b\}$$

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$$

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We write $|w|$ for the **length** of a word w .

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$$\Sigma = \{a, b\}$$

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, \dots\}$$

$$|aba| = 3, |b| = 1, |\varepsilon| = 0$$

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A **formal language** (over alphabet Σ) is a subset of Σ^* .

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Example

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Languages: Examples

Example (Languages over $\Sigma = \{a, b\}$)

- $S_1 = \{a, aa, aaa, aaaa, \dots\}$

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- $S_4 = \{\varepsilon\}$
- $S_5 = \emptyset$

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- $S_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many } a\text{'s as } b\text{'s}\}$
 $= \{\varepsilon, aab, aba, baa, \dots\}$

Languages: Examples

Example (Languages over $\Sigma = \{a, b\}$)

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- $S_6 = \{w \in \Sigma^* \mid w \text{ contains twice as many } a\text{'s as } b\text{'s}\}$
 $= \{\varepsilon, aab, aba, baa, \dots\}$
- $S_7 = \{w \in \Sigma^* \mid |w| = 3\}$
 $= \{aaa, aab, aba, baa, bba, bab, abb, bbb\}$

Questions



Questions?

Grammars

Grammars

Definition (Grammars)

A **grammar** is a 4-tuple $\langle \Sigma, V, P, S \rangle$ with:

- 1 Σ finite alphabet of **terminal symbols**
- 2 V finite set of **variables (nonterminal symbols)**
with $V \cap \Sigma = \emptyset$
- 3 $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ finite set of **rules** (or productions)
- 4 $S \in V$ **start variable**

German: Grammatik, Terminalalphabet, Variablen,
Regeln/Produktionen, Startvariable

Rule Sets

What exactly does $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ mean?

- $(V \cup \Sigma)^*$: all words over $(V \cup \Sigma)$
- $(V \cup \Sigma)^+$: all non-empty words over $(V \cup \Sigma)$
in general, for set X : $X^+ = X^* \setminus \{\varepsilon\}$
- \times : Cartesian product
- $(V \cup \Sigma)^+ \times (V \cup \Sigma)^*$: set of all pairs $\langle x, y \rangle$, where x non-empty word over $(V \cup \Sigma)$ and y word over $(V \cup \Sigma)$
- Instead of $\langle x, y \rangle$ we usually write rules in the form $x \rightarrow y$.

Rules: Examples

Example

Let $\Sigma = \{a, b, c\}$ and $V = \{X, Y, Z\}$.

The following rules are in $(V \cup \Sigma)^+ \times (V \cup \Sigma)^*$:

$$X \rightarrow XaY$$

$$Yb \rightarrow a$$

$$XY \rightarrow \varepsilon$$

$$XYZ \rightarrow abc$$

$$abc \rightarrow XYZ$$

Derivations

Definition (Derivations)

Let $\langle \Sigma, V, P, S \rangle$ be a grammar. A word $v \in (V \cup \Sigma)^*$ can be **derived** from word $u \in (V \cup \Sigma)^+$ (written as $u \Rightarrow v$) if

- 1 $u = xyz$, $v = xy'z$ with $x, z \in (V \cup \Sigma)^*$ and
- 2 there is a rule $y \rightarrow y' \in P$.

We write: $u \Rightarrow^* v$ if v can be derived from u in finitely many steps (i. e., by using n rules for $n \in \mathbb{N}_0$).

German: Ableitung

Language Generated by a Grammar

Definition (Languages)

The **language generated** by a grammar $G = \langle \Sigma, V, P, S \rangle$

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

is the set of all words from Σ^* that can be derived from S with finitely many rule applications.

German: erzeugte Sprache

Grammars

Examples: blackboard

Questions



Questions?

Chomsky Hierarchy

Chomsky Hierarchy

Grammars are ordered into the **Chomsky hierarchy**.

Definition (Chomsky Hierarchy)

- Every grammar is of **type 0** (all rules allowed).
- Grammar is of **type 1** (**context-sensitive**) if all rules $w_1 \rightarrow w_2$ satisfy $|w_1| \leq |w_2|$.
- Grammar is of **type 2** (**context-free**) if additionally $w_1 \in V$ (single variable) in all rules $w_1 \rightarrow w_2$.
- Grammar is of **type 3** (**regular**) if additionally $w_2 \in \Sigma \cup \Sigma V$ in all rules $w_1 \rightarrow w_2$.

special case: rule $S \rightarrow \varepsilon$ is always allowed if S is the start variable and never occurs on the right-hand side of any rule.

German: Chomsky-Hierarchie, Typ 0, Typ 1 (kontextsensitiv), Typ 2 (kontextfrei), Typ 3 (regulär)

Chomsky Hierarchy

Examples: blackboard

Chomsky Hierarchy

Definition (Type 0–3 Languages)

A language $L \subseteq \Sigma^*$ is of type 0 (type 1, type 2, type 3) if there exists a type-0 (type-1, type-2, type-3) grammar G with $\mathcal{L}(G) = L$.

Type k Language: Example

Example

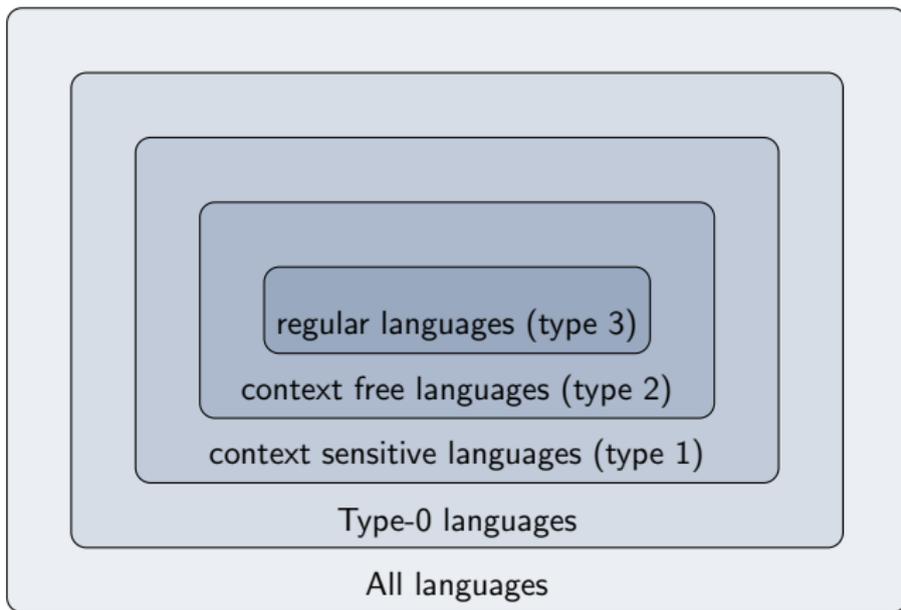
Consider the language L generated by the grammar $\langle \{a, b, c, \neg, \wedge, \vee, (,)\}, \{F, A, N, C, D\}, P, F \rangle$ with the following rules P :

$$\begin{array}{lll} F \rightarrow A & A \rightarrow a & N \rightarrow \neg F \\ F \rightarrow N & A \rightarrow b & C \rightarrow (F \wedge F) \\ F \rightarrow C & A \rightarrow c & D \rightarrow (F \vee F) \\ F \rightarrow D & & \end{array}$$

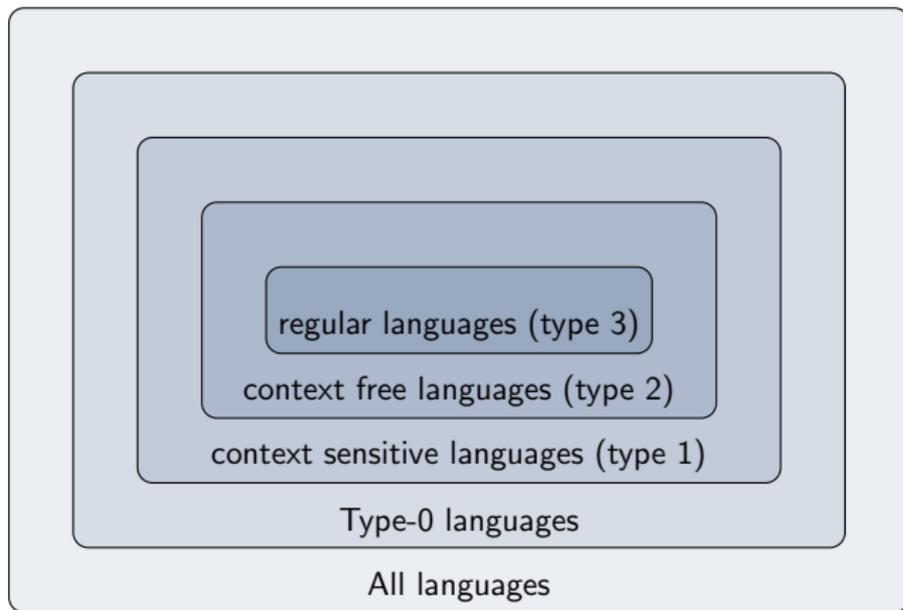
Questions:

- Is L a type-0 language?
- Is L a type-1 language?
- Is L a type-2 language?
- Is L a type-3 language?

Chomsky Hierarchy



Chomsky Hierarchy



Note: Not all languages can be described by grammars. ([Proof?](#))

Questions



Questions?

Summary

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- **Languages** are sets of symbol sequences.
- **Grammars** are one possible way to specify languages.
- Language **generated** by a grammar is the set of all words (of nonterminal symbols) **derivable** from the start symbol.
- **Chomsky hierarchy** distinguishes between languages at different levels of expressiveness.

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next chapters:

- more about regular languages
- automata as alternative representation of languages