

Theory of Computer Science

B3. Predicate Logic I

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Motivation

Limits of Propositional Logic

group axioms for $\langle G, \circ \rangle$ and neutral element e :

- 1 For all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$.
- 2 For all $x \in G$, $x \circ e = x$.
- 3 For all $x \in G$, there is a $y \in G$ with $x \circ y = e$.

Limits of Propositional Logic

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cannot be expressed in propositional logic:

- objects x, y, z from G
- object references $(x \circ y)$ with function $\circ$
- equality $=$
- “for all”, “there is”

Limits of Propositional Logic

group axioms for $\langle G, \circ \rangle$ and neutral element e :

- ① For all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$.
- ② For all $x \in G$, $x \circ e = x$.
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cannot be expressed in propositional logic:

- objects x, y, z from G
- object references $(x \circ y)$ with function $\circ$
- equality $=$
- “for all”, “there is”

▷ need more expressive logic

↪ predicate logic

German: Prädikatenlogik

Syntax of Predicate Logic

Syntax: Building Blocks

- **Signatures** define allowed symbols.
analogy: variable set A in propositional logic
- **Terms** are associated with objects by the semantics.
no analogy in propositional logic
- **Formulas** are associated with truth values (*true* or *false*) by the semantics.
analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

Definition (Signature)

A **signature** (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ consisting of the following four disjoint sets:

- a finite or countable set $\mathcal{V}$ of **variable symbols**
- a finite or countable set $\mathcal{C}$ of **constant symbols**
- a finite or countable set $\mathcal{F}$ of **function symbols**
- a finite or countable set $\mathcal{P}$ of **predicate symbols**
(or **relation symbols**)

Every function symbol $f \in \mathcal{F}$ and predicate symbol $P \in \mathcal{P}$ has an associated **arity** $ar(f), ar(P) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:

- *k*-ary (function or predicate) symbol:
symbol s with arity $ar(s) = k$.
- also: unary, binary, ternary

German: k -stellig, unär, binär, ternär

conventions (in this lecture):

- variable symbols written in *italics*,
other symbols upright.
- predicate symbols begin with capital letter,
other symbols with lower-case letters

Signatures: Examples

Example: Arithmetic

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{zero}, \text{one}\}$
- $\mathcal{F} = \{\text{sum}, \text{product}\}$
- $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$

$ar(\text{sum}) = ar(\text{product}) = 2, ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

Signatures: Examples

Example: Genealogy

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{roger-federer, lisa-simpson}\}$
- $\mathcal{F} = \emptyset$
- $\mathcal{P} = \{\text{Female, Male, Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1, ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

A **term** (over $\mathcal{S}$) is inductively constructed according to the following rules:

- Every variable symbol $v \in \mathcal{V}$ is a term.
- Every constant symbol $c \in \mathcal{C}$ is a term.
- If $t_1, \dots, t_k$ are terms and $f \in \mathcal{F}$ is a function symbol with arity k , then $f(t_1, \dots, t_k)$ is a term.

German: Term

Terms: Definition

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German: Term

examples:

- x_4
- lisa-simpson
- $\text{sum}(x_3, \text{product}(\text{one}, x_5))$

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

- If $t_1, \dots, t_k$ are terms (over $\mathcal{S}$) and $P \in \mathcal{P}$ is a k -ary predicate symbol, then the **atomic formula** (or the **atom**) $P(t_1, \dots, t_k)$ is a formula over $\mathcal{S}$.
- If t_1 and t_2 are terms (over $\mathcal{S}$), then the **identity** $(t_1 = t_2)$ is a formula over $\mathcal{S}$.
- If $x \in \mathcal{V}$ is a variable symbol and φ a formula over $\mathcal{S}$, then the **universal quantification** $\forall x \varphi$ and the **existential quantification** $\exists x \varphi$ are formulas over $\mathcal{S}$.

...

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

...

- If φ is a formula over $\mathcal{S}$, then so is its **negation** $\neg\varphi$.
- If φ and ψ are formulas over $\mathcal{S}$, then so are the **conjunction** $(\varphi \wedge \psi)$ and the **disjunction** $(\varphi \vee \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

Examples: Arithmetic and Genealogy

- $\text{Positive}(x_2)$
- $\forall x (\neg \text{SquareNumber}(x) \vee \text{Positive}(x))$
- $\exists x_3 (\text{SquareNumber}(x_3) \wedge \neg \text{Positive}(x_3))$
- $\forall x (x = y)$
- $\forall x (\text{sum}(x, x) = \text{product}(x, \text{one}))$
- $\forall x \exists y (\text{sum}(x, y) = \text{zero})$
- $\forall x \exists y (\text{Parent}(y, x) \wedge \text{Female}(y))$

Terminology: The symbols $\forall$ and $\exists$ are called **quantifiers**.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:

- $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
- $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.
- Sequences of the same quantifier can be abbreviated.

For example:

- $\forall x\forall y\forall z \varphi \rightsquigarrow \forall xyz \varphi$
- $\exists x\exists y\exists z \varphi \rightsquigarrow \exists xyz \varphi$
- $\forall w\exists x\exists y\forall z \varphi \rightsquigarrow \forall w\exists xy\forall z \varphi$

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- $\forall x \forall y \forall z \varphi \rightsquigarrow \forall xyz \varphi$
- $\exists x \exists y \exists z \varphi \rightsquigarrow \exists xyz \varphi$
- $\forall w \forall x \exists y \forall z \varphi \rightsquigarrow \forall w \exists xy \forall z \varphi$

placement of parentheses by convention:

- analogous to propositional logic
- quantifiers $\forall$ and $\exists$ bind more strongly than anything else.
- **example:** $\forall x P(x) \rightarrow Q(x)$ corresponds to $(\forall x P(x) \rightarrow Q(x))$,
not $\forall x (P(x) \rightarrow Q(x))$.

Questions



Questions?

Semantics of Predicate Logic

Semantics: Motivation

- interpretations in propositional logic:
truth assignments for the **propositional variables**
- There are no propositional variables in predicate logic.
- instead: interpretation determines meaning
of the **constant**, **function** and **predicate symbols**.
- meaning of **variable symbols** not determined by interpretation
but by separate **variable assignment**.

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An **interpretation** (for $\mathcal{S}$) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- a non-empty set U called the **universe** and
- a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant, function, and predicate symbols:
 - $c^{\mathcal{I}} \in U$ for constant symbols $c \in \mathcal{C}$
 - $f^{\mathcal{I}} : U^k \rightarrow U$ for k -ary function symbols $f \in \mathcal{F}$
 - $P^{\mathcal{I}} \subseteq U^k$ for k -ary predicate symbols $P \in \mathcal{P}$

A **variable assignment** (for $\mathcal{S}$ and universe U) is a function $\alpha : \mathcal{V} \rightarrow U$.

German: Interpretation, Variablenzuweisung, Universum (or Grundmenge)

Interpretations and Variable Assignments: Example

Example

signature: $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
 $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$, $\mathcal{P} = \{\text{SquareNumber}\}$
 $ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{SquareNumber}) = 1$

Interpretations and Variable Assignments: Example

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$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
- $\text{one}^{\mathcal{I}} = u_1$
- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x(\text{Block}(x) \rightarrow \text{Red}(x)) \wedge \text{Block}(a))$

“For all objects x : if x is a block, then x is red.

Also, the object called a is a block.”

- **Terms** are interpreted as **objects**.
- **Unary predicates** denote properties of objects
(to be a block, to be red, to be a square number, ...)
- **General predicates** denote relations between objects
(to be someone's child, to have a common divisor, ...)
- **Universally quantified** formulas (“ $\forall$ ”) are true
if they hold for **every** object in the universe.
- **Existentially quantified** formulas (“ $\exists$ ”) are true
if they hold for **at least one** object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U .

Let t be a term over $\mathcal{S}$.

The **interpretation of t** under $\mathcal{I}$ and α , written as $t^{\mathcal{I}, \alpha}$,
is the element of the universe U defined as follows:

- If $t = x$ with $x \in \mathcal{V}$ (t is a **variable term**):
 $x^{\mathcal{I}, \alpha} = \alpha(x)$
- If $t = c$ with $c \in \mathcal{C}$ (t is a **constant term**):
 $c^{\mathcal{I}, \alpha} = c^{\mathcal{I}}$
- If $t = f(t_1, \dots, t_k)$ (t is a **function term**):
 $f(t_1, \dots, t_k)^{\mathcal{I}, \alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$,

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Interpretations of Terms: Example

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$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
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- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
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$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

Example (ctd.)

- $\text{zero}^{\mathcal{I}, \alpha} =$
- $y^{\mathcal{I}, \alpha} =$
- $\text{sum}(x, y)^{\mathcal{I}, \alpha} =$
- $\text{product}(\text{one}, \text{sum}(x, \text{zero}))^{\mathcal{I}, \alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U .
We say that $\mathcal{I}$ and α **satisfy** a predicate logic formula φ
(also: φ is **true** under $\mathcal{I}$ and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$\mathcal{I}, \alpha \models P(t_1, \dots, t_k) \quad \text{iff } \langle t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$$

$$\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff } t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$$

$$\mathcal{I}, \alpha \models \neg \varphi \quad \text{iff } \mathcal{I}, \alpha \not\models \varphi$$

$$\mathcal{I}, \alpha \models (\varphi \wedge \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi$$

$$\mathcal{I}, \alpha \models (\varphi \vee \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi$$

...

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

...

$$\mathcal{I}, \alpha \models \forall x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U$$

$$\mathcal{I}, \alpha \models \exists x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U$$

where $\alpha[x := u]$ is the same variable assignment as α , except that it maps variable x to the value u .

Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

German: $\mathcal{I}$ und α erfüllen φ (also: φ ist wahr unter $\mathcal{I}$ und α)

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{a, b\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\text{Block}, \text{Red}\}$,

$ar(\text{Block}) = ar(\text{Red}) = 1$.

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{a, b\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\text{Block}, \text{Red}\}$,
 $ar(\text{Block}) = ar(\text{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- $a^{\mathcal{I}} = u_1$
- $b^{\mathcal{I}} = u_3$
- $\text{Block}^{\mathcal{I}} = \{u_1, u_2\}$
- $\text{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\text{Block}(b) \vee \neg \text{Block}(b))?$
- $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg \text{Block}(y)))?$
- $\mathcal{I}, \alpha \models (\text{Block}(a) \wedge \text{Block}(b))?$
- $\mathcal{I}, \alpha \models \forall x(\text{Block}(x) \rightarrow \text{Red}(x))?$

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\text{Block}(b) \vee \neg \text{Block}(b))$?

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg \text{Block}(y)))$?

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\text{Block}(a) \wedge \text{Block}(b))$?

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models \forall x(\text{Block}(x) \rightarrow \text{Red}(x))$?

Questions



Questions?

Summary

Summary

- **Predicate logic** is more expressive than propositional logic and allows statements over **objects** and their **properties**.
- Objects are described by **terms** that are built from variable, constant and function symbols.
- Properties and relations are described by **formulas** that are built from predicates, quantifiers and the usual logical operators.
- As with all logics, we analyze
 - **syntax**: what is a formula?
 - **semantics**: how do we interpret a formula?