Theory of Computer Science
B1. Propositional Logic I

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Motivation

Exercise from Last Lecture

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Simplify this advice!

Exercise by U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Why Logic?

- formalizing mathematics
  - What is a true statement?
  - What is a valid proof?
- basis of many tools in computer science
  - design of digital circuits
  - meaning of programming languages
  - semantics of databases; query optimization
  - verification of safety-critical hardware/software
  - knowledge representation in artificial intelligence
  - . . .

Example: Group Theory

Example of a group (in mathematics): $\langle \mathbb{Z}, + \rangle$

- the set of integers with the addition operation

A group in general: $\langle G, \circ \rangle$

- G is a set and $\circ : G \times G \to G$ is called the group operation; we write "$x \circ y$" instead of "$\circ(x, y)$" (infix notation)

Example: Group Theory

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For $\langle G, \circ \rangle$ to be a group, it must satisfy the group axioms:

- (G1) For all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$.
- There exists $e \in G$ (called the neutral element) such that:
  - (G2) for all $x \in G$, $x \circ e = x$, and
  - (G3) for all $x \in G$, there is a $y \in G$ with $x \circ y = e$.

German: Gruppe, Verknüpfung, Infix, Gruppenaxiome, neutrales Element

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Properties of Propositional Formulas
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Example: Group Theory

Theorem (Existence of a left inverse)

Let $\langle G, \circ \rangle$ be a group with neutral element e.
For all $x \in G$ there is a $y \in G$ with $y \circ x = e$.

Proof.

Consider an arbitrary $x \in G$.
Because of G3, there is a y with $x \circ y = e$ (*).
Also because of G3, for this y there is a z with $y \circ z = e$ (**).
It follows that:

$$
\begin{aligned}
y \circ x &\overset{(G2)}{=} (y \circ x) \circ e \overset{(**)}{=} (y \circ x) \circ (y \circ z) \\
&\overset{(G1)}{=} y \circ (x \circ (y \circ z)) \overset{(G1)}{=} y \circ ((x \circ y) \circ z) \\
&\overset{(*)}{=} y \circ (e \circ z) \overset{(G1)}{=} (y \circ e) \circ z \\
&\overset{(G2)}{=} y \circ z \overset{(**)}{=} e
\end{aligned}
$$

$\square$

What Logic is About

General Question:

- Given a set of axioms (e. g., group axioms)
- what can we derive from them?
  (e. g., theorem about the existence of a left inverse)
- And on what basis may we argue?
  (e. g., why does $y \circ x = (y \circ x) \circ e$ follow from axiom G2?)

⤳ logic

Goal: "mechanical" proofs

- formal "game with letters"
- detached from a concrete meaning

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Propositional Logic

Propositional logic is a simple logic without numbers or objects.

Building blocks of propositional logic:

- propositions are statements that can be either true or false
- atomic propositions cannot be split into sub-propositions
- logical connectives connect propositions to form new ones

German: Aussagenlogik, Aussage, atomare Aussage, Junktoren

Examples for Building Blocks

If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- Every sentence is a proposition that consists of sub-propositions (e. g., "eat ice cream or don't drink beer").

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Examples for Building Blocks

If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- Every sentence is a proposition that consists of sub-propositions (e. g., "eat ice cream or don't drink beer").
- atomic propositions "drink beer", "eat fish", "eat ice cream"

Examples for Building Blocks

If I **don't** drink beer to a meal, **then** I always eat fish. **Whenever** I have fish **and** beer with the same meal, I **abstain** from ice cream. **When** I eat ice cream **or** **don't** drink beer, **then** I **never** touch fish.

- Every sentence is a proposition that consists of sub-propositions (e. g., "eat ice cream or don't drink beer").
- atomic propositions "drink beer", "eat fish", "eat ice cream"
- logical connectives "and", "or", negation, "if, then"

Problems with Natural Language

If I don't drink beer to a meal, then I always eat fish.
Whenever I have fish and beer with the same meal, I abstain from ice cream.
When I eat ice cream or don't drink beer, then I never touch fish.

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- "irrelevant" information

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Problems with Natural Language

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- "irrelevant" information
- different formulations for the same connective/proposition

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Problems with Natural Language

If I don't drink beer, then I eat fish.
Whenever I have fish and beer, I abstain from ice cream.
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- "irrelevant" information
- different formulations for the same connective/proposition

Problems with Natural Language

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

- "irrelevant" information
- different formulations for the same connective/proposition

Exercise by U. Schöning: Logik für Informatiker
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What is Next?

- What are meaningful (well-defined) sequences of atomic propositions and connectives?
  "if then EatIceCream not or DrinkBeer and" not meaningful
  → syntax

- What does it mean if we say that a statement is true?
  Is "DrinkBeer and EatFish" true?
  → semantics

- When does a statement logically follow from another?
  Does "EatFish" follow from "if DrinkBeer, then EatFish"?
  → logical entailment

- German: Syntax, Semantik, logische Folgerung

Questions

Questions?

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Syntax

Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of atomic propositions. The set of propositional formulas (over A) is inductively defined as follows:

- Every atom $a \in A$ is a propositional formula over A.
- If φ is a propositional formula over A,
  then so is its negation $\neg\varphi$.
- If φ and ψ are propositional formulas over A,
  then so is the conjunction $(\varphi \wedge \psi)$.
- If φ and ψ are propositional formulas over A,
  then so is the disjunction $(\varphi \vee \psi)$.

The implication $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.
German: atomare Aussage, aussagenlogische Formel, Atom,
Negation, Konjunktion, Disjunktion, Implikation, Bikonditional

Syntax: Examples

Which of the following sequences of symbols are propositional
formulas over the set of all possible letter sequences?

- $(A \land (B \lor C))$

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Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences?

- $(A \land (B \lor C))$
- $((\text{EatFish} \land \text{DrinkBeer}) \to \neg \text{EatIceCream})$

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences?

- $(A \wedge (B \vee C))$
- $((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg\text{EatIceCream})$
- $\neg(\wedge \text{Rain} \vee \text{StreetWet})$

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- $\neg(A = B)$

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- $((\text{EatFish} \land \text{DrinkBeer}) \to \neg \text{EatIceCream})$
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- $\neg(\text{Rain} \lor \text{StreetWet})$
- $\neg(A = B)$
- $(A \land \neg(B \leftrightarrow)C)$

Syntax: Examples

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- $(A \lor \neg(B \leftrightarrow C))$

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- $\neg(A = B)$
- $(A \wedge \neg(B \leftrightarrow)C)$
- $(A \vee \neg(B \leftrightarrow C))$
- $((A \leq B) \wedge C)$

Syntax: Examples

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- $(A \wedge \neg(B \leftrightarrow)C)$
- $(A \vee \neg(B \leftrightarrow C))$
- $((A \leq B) \wedge C)$
- $((A_1 \wedge A_2) \vee \neg(A_3 \leftrightarrow A_2))$

Syntax: Examples

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- $(A \land (B \lor C))$
- $((EatFish \land DrinkBeer) \to \neg EatIceCream)$
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- $\neg(Rain \lor StreetWet)$
- $\neg(A = B)$
- $(A \land \neg(B \leftrightarrow)C)$
- $(A \lor \neg(B \leftrightarrow C))$
- $((A \leq B) \land C)$
- $((A_1 \land A_2) \lor \neg(A_3 \leftrightarrow A_2))$

Which kinds of formula are they (atom, conjunction, . . .)?

Questions

Questions?

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Semantics

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:
((EatFish $\land$ DrinkBeer) $\rightarrow$ ¬EatIceCream)?

▷ We need semantics!

Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of atomic propositions A is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula φ (over A) holds under $\mathcal{I}$
(written as $\mathcal{I} \models \varphi$) according to the following definition:

$$
\begin{array}{lll}
\mathcal{I} \models a \ (\text{for } a \in A) & \text{iff} & \mathcal{I}(a) = 1 \\
\mathcal{I} \models \neg\varphi & \text{iff} & \text{not } \mathcal{I} \models \varphi \\
\mathcal{I} \models (\varphi \wedge \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \psi \\
\mathcal{I} \models (\varphi \vee \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \psi
\end{array}
$$

Question: should we define semantics of $(\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$?

German: Wahrheitsbelegung/Interpretation, φ gilt unter $\mathcal{I}$

Semantics of Propositional Logic: Terminology

- For $\mathcal{I} \models \varphi$ we also say $\mathcal{I}$ is a model of φ
  and that φ is true under $\mathcal{I}$.

- If φ does not hold under $\mathcal{I}$, we write this as $\mathcal{I} \not\models \varphi$
  and say that $\mathcal{I}$ is no model of φ
  and that φ is false under $\mathcal{I}$.

- Note: $\models$ is not part of the formula
  but part of the meta language (speaking about a formula).

German: $\mathcal{I}$ ist ein/kein Modell von φ; φ ist wahr/falsch unter $\mathcal{I}$;
Metasprache

Semantics: Example (1)

$A = \{\text{DrinkBeer}, \text{EatFish}, \text{EatIceCream}\}$

$\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$

$\varphi = (\neg \text{DrinkBeer} \rightarrow \text{EatFish})$

Do we have $\mathcal{I} \models \varphi$?

Semantics: Example (2)

Goal: prove $\mathcal{I} \models \varphi$.

Let us use the definitions we have seen:

$$
\begin{aligned}
\mathcal{I} \models \varphi \text{ iff } & \mathcal{I} \models (\neg \text{DrinkBeer} \rightarrow \text{EatFish}) \\
\text{iff } & \mathcal{I} \models (\neg\neg \text{DrinkBeer} \vee \text{EatFish}) \\
\text{iff } & \mathcal{I} \models \neg\neg \text{DrinkBeer or } \mathcal{I} \models \text{EatFish}
\end{aligned}
$$

This means that if we want to prove $\mathcal{I} \models \varphi$, it is sufficient to prove

$$\mathcal{I} \models \neg\neg \text{DrinkBeer}$$

or to prove

$$\mathcal{I} \models \text{EatFish}.$$

We attempt to prove the first of these statements.

Semantics: Example (3)

New goal: prove $\mathcal{I} \models \neg\neg\text{DrinkBeer}$.

We again use the definitions:

$$\begin{aligned}
\mathcal{I} \models \neg\neg\text{DrinkBeer} \;&\text{iff not } \mathcal{I} \models \neg\text{DrinkBeer} \\
&\text{iff not not } \mathcal{I} \models \text{DrinkBeer} \\
&\text{iff } \mathcal{I} \models \text{DrinkBeer} \\
&\text{iff } \mathcal{I}(\text{DrinkBeer}) = 1
\end{aligned}$$

The last statement is true for our interpretation $\mathcal{I}$.

To write this up as a proof of $\mathcal{I} \models \varphi$,
we can go through this line of reasoning back-to-front,
starting from our assumptions and ending with the conclusion
we want to show.

Semantics: Example (4)

Let $\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$.

Proof that $\mathcal{I} \models (\neg\text{DrinkBeer} \to \text{EatFish})$:

(1) We have $\mathcal{I} \models \text{DrinkBeer}$
    (uses defn. of $\models$ for atomic props. and fact $\mathcal{I}(\text{DrinkBeer}) = 1$).

(2) From (1), we get $\mathcal{I} \not\models \neg\text{DrinkBeer}$
    (uses defn. of $\models$ for negations).

(3) From (2), we get $\mathcal{I} \models \neg\neg\text{DrinkBeer}$
    (uses defn. of $\models$ for negations).

(4) From (3), we get $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \lor \psi)$ for all formulas ψ,
    in particular $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \lor \text{EatFish})$
    (uses defn. of $\models$ for disjunctions).

(5) From (4), we get $\mathcal{I} \models (\neg\text{DrinkBeer} \to \text{EatFish})$
    (uses defn. of "$\to$"). $\qquad\square$

Questions

Questions?

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Properties of Propositional Formulas
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Properties of Propositional Formulas

Properties of Propositional Formulas

A propositional formula φ is

- satisfiable if φ has at least one model
- unsatisfiable if φ is not satisfiable
- valid (or a tautology) if φ is true under every interpretation
- falsifiable if φ is no tautology

German: erfüllbar, unerfüllbar, gültig/eine Tautologie, falsifizierbar

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How can we show that a formula has one of these properties?

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Properties of Propositional Formulas
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Examples

- Show that (A ∧ B) is satisfiable.

Examples

- Show that $(A \wedge B)$ is satisfiable.

  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \wedge B)$)

Examples

- Show that $(A \land B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \land B)$)
- Show that $(A \land B)$ is falsifiable.

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- Show that $(A \wedge B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \wedge B)$)
- Show that $(A \wedge B)$ is falsifiable.
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- Show that $(A \wedge B)$ is not valid.

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Properties of Propositional Formulas
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Examples

- Show that $(A \land B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ ($+$ simple proof that $\mathcal{I} \models (A \land B)$)
- Show that $(A \land B)$ is falsifiable.
  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$ ($+$ simple proof that $\mathcal{I} \not\models (A \land B)$)
- Show that $(A \land B)$ is not valid.
  Follows directly from falsifiability.

Examples

- Show that $(A \land B)$ is satisfiable.
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- Show that $(A \land B)$ is not unsatisfiable.

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  Follows directly from satisfiability.

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  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \land B)$)

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  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \not\models (A \land B)$)

- Show that $(A \land B)$ is not valid.
  Follows directly from falsifiability.

- Show that $(A \land B)$ is not unsatisfiable.
  Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/
not satisfiable/not falsifiable?

Examples

- Show that $(A \wedge B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \wedge B)$)

- Show that $(A \wedge B)$ is falsifiable.
  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \not\models (A \wedge B)$)

- Show that $(A \wedge B)$ is not valid.
  Follows directly from falsifiability.

- Show that $(A \wedge B)$ is not unsatisfiable.
  Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/ not satisfiable/not falsifiable?

⤳ must consider all possible interpretations

Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

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if they are models of the considered formula.

$$\begin{array}{c|c} \mathcal{I}(A) & \mathcal{I} \models \neg A \\ \hline 0 & \\ 1 & \end{array}$$

Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

$\mathcal{I}(A)$	$\mathcal{I} \models \neg A$
0	Yes
1	No

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0	Yes
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$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \wedge B)$
0	0	
0	1	
1	0	
1	1	

Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

$$\begin{array}{c|c} \mathcal{I}(A) & \mathcal{I} \models \neg A \\ \hline 0 & \text{Yes} \\ 1 & \text{No} \end{array}$$

$$\begin{array}{cc|c} \mathcal{I}(A) & \mathcal{I}(B) & \mathcal{I} \models (A \wedge B) \\ \hline 0 & 0 & \text{No} \\ 0 & 1 & \text{No} \\ 1 & 0 & \text{No} \\ 1 & 1 & \text{Yes} \end{array}$$

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if they are models of the considered formula.

$\mathcal{I}(A)$	$\mathcal{I} \models \neg A$
0	Yes
1	No

$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \wedge B)$		$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \vee B)$
0	0	No		0	0	No
0	1	No		0	1	Yes
1	0	No		1	0	Yes
1	1	Yes		1	1	Yes

Truth Tables in General

Similarly in the case where we consider a formula whose building
blocks are themselves arbitrary unspecified formulas:

$\mathcal{I} \models \varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models (\varphi \wedge \psi)$
No	No	No
No	Yes	No
Yes	No	No
Yes	Yes	Yes

Truth Tables in General

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No	No	No
No	Yes	No
Yes	No	No
Yes	Yes	Yes

Exercises: truth table for $(\varphi \rightarrow \psi)$

Truth Tables for Properties of Formulas

Is $\varphi = ((A \rightarrow B) \vee (\neg B \rightarrow A))$ valid, unsatisfiable, ...?

$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models \neg B$	$\mathcal{I} \models (A \rightarrow B)$	$\mathcal{I} \models (\neg B \rightarrow A)$	$\mathcal{I} \models \varphi$
0	0	Yes	Yes	No	Yes
0	1	No	Yes	Yes	Yes
1	0	Yes	No	Yes	Yes
1	1	No	Yes	Yes	Yes

Connection Between Formula Properties and Truth Tables

A propositional formula φ is

- **satisfiable** if φ has at least one model
  $\rightsquigarrow$ result in at least one row is "Yes"

- **unsatisfiable** if φ is not satisfiable
  $\rightsquigarrow$ result in all rows is "No"

- **valid** (or a **tautology**) if φ is true under every interpretation
  $\rightsquigarrow$ result in all rows is "Yes"

- **falsifiable** if φ is no tautology
  $\rightsquigarrow$ result in at least one row is "No"

Main Disadvantage of Truth Tables

How big is a truth table with n atomic propositions?

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⤳ not viable for larger formulas; we need a different solution
⤳ Foundations of Artificial Intelligence course

Motivation
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Syntax
oooo

Semantics
ooooooooo

Properties of Propositional Formulas
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Summary
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Questions

Questions?

Motivation
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Syntax
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Semantics
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Properties of Propositional Formulas
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Summary
●○

Summary

Summary

- propositional logic based on atomic propositions
- syntax defines what well-formed formulas are
- semantics defines when a formula is true
- interpretations are the basis of semantics
- satisfiability and validity are important properties of formulas
- truth tables systematically consider all possible interpretations
- truth tables are only useful for small formulas