Theory of Computer Science

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Spring Term 2016

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Computer Science

Exercise Sheet 1

Due: Wednesday, March 2, 2016

Note: Submissions that are exclusively created with LaTeX will receive a bonus mark. Please submit only the resulting PDF file (or a printout of this file).

Note: The goal of this exercise is to learn how to correctly express formal proofs. A formally correct proof consists of single steps where each step follows *immediately* from the previous steps or from the assumptions (for example when replacing a value by its definition). Please write down your proofs in detail and in a formal fashion. Examples can be fould in the lecture slides.

Exercise 1.1 (Direct Proof; 1.5 + 1.5 Points)

(a) Prove the following statement for any sets A, B, and C with a direct proof:

$$((A \cap B) \cap C) = (A \cap (B \cap C))$$

(b) Refute the claim that the following statement holds for any sets A, B, and C by specifying a *counter example* (i.e., sets A, B, and C) and showing that it does not satisfy the statement.

$$((A \setminus B) \setminus C) = (A \setminus (B \setminus C))$$

Exercise 1.2 (Proof by Contradiction; 2 Points)

Prove by contradiction that for all $n \in \mathbb{N}_0$ the following holds: if $n + 7$ is prime, then n is not prime.

Hint: 2 is the only even prime number.

Exercise 1.3 (Mathematical Induction; 1 Point)

We want to prove that all leopards have the same speed. Which line of the following proof by mathematical induction has an error? What exactly is the error on the formal level?

1. Since we do not know the exact number of leopards we will prove that for any set of leopards all leopards in the set have the same speed.
2. We define the property $L(n)$: In any set of n leopards, all leopards have the same speed.
3. As the induction basis we show $L(1)$.
4. In a set consisting of one leopard all leopards in the set have the same speed. Thus $L(1)$ holds.
5. The induction hypothesis is: $L(i)$ holds for all $1 \leq i \leq n$.
6. We need to show $L(n + 1)$ under the assumption of the induction hypothesis.
7. We consider a set of $n + 1$ leopards $M = \{l_1, \dots, l_n, l_{n+1}\}$
8. The set $M_1 = M \setminus \{l_1\} = \{l_2, \dots, l_{n+1}\}$ (i.e., M without the first leopard) contains n leopards ($|M_1| = n$).

9. According to the induction hypothesis $L(n)$ holds for M_1 and thus we can conclude that all leopards in M_1 have the same speed.
10. The set $M_2 = M \setminus \{l_{n+1}\} = \{l_1, \ldots, l_n\}$ (i.e., M without the last leopard) contains n leopards ($|M_2| = n$).
11. According to the induction hypothesis $L(n)$ holds for M_2 and thus we can conclude that all leopards in M_2 have the same speed.
12. Now we consider a leopard l_M which is contained in both sets, for example $l_M = l_2$.
13. l_M has the same speed as all leopards from M_1 and also as all leopards from M_2.
14. From this we conclude that all leopards from $M_1 \cup M_2 = M$ must have the same speed, which means we have shown that $L(n+1)$ holds for M.
15. Since $L(1)$ holds, and, under hypothesis $L(n)$, $L(n+1)$ also holds for $n \geq 1$, we conclude that $L(n)$ holds for all $n \in \mathbb{N}$.
16. In particular, $L(n)$ holds for the set of all leopards, which means that all leopards have the same speed.

Exercise 1.4 (Structural Induction; 2 + 2 Points)

(a) We inductively define a set of simple mathematical expressions which only utilize the following symbols: "Z", "T", "$\oplus$", "$\otimes$", "$\llbracket$", and "$\rrbracket$". The set $\mathcal{E}$ of *simple expressions* is inductively defined as follows:

- Z and T are simple expressions.
- If x and y are simple expressions, $\llbracket x \otimes y \rrbracket$ is also a simple expression.
- If x and y are simple expressions, $\llbracket x \oplus y \rrbracket$ is also a simple expression.

Examples for simple expressions: T, $\llbracket \mathtt{T} \otimes \mathtt{Z} \rrbracket$, $\llbracket \llbracket \mathtt{T} \otimes \mathtt{T} \rrbracket \oplus \llbracket \mathtt{Z} \oplus \mathtt{T} \rrbracket \rrbracket$

Furthermore we define a function $f : \mathcal{E} \to \mathbb{N}_0$ as follows:

- $f(\mathtt{Z}) = 0$, $f(\mathtt{T}) = 2$
- $f(\llbracket x \otimes y \rrbracket) = f(x) \cdot f(y)$
- $f(\llbracket x \oplus y \rrbracket) = f(x) + f(y)$

So for example: $f(\mathtt{T}) = 2$, $f(\llbracket \mathtt{T} \otimes \mathtt{Z} \rrbracket) = f(\mathtt{T}) \cdot f(\mathtt{Z}) = 2 \cdot 0 = 0$, $f(\llbracket \llbracket \mathtt{T} \otimes \mathtt{T} \rrbracket \oplus \llbracket \mathtt{Z} \oplus \mathtt{T} \rrbracket \rrbracket) = 6$.

Prove the following property for all simple expressions $x \in \mathcal{E}$ by structural induction:

$$f(x) \text{ is even.}$$

(b) We define two functions over binary trees (as presented in the lecture): $height : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to its height and $leaves : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to the number of its leaves.

- $height(\Box) = 1$
- $height(\langle B_L, \bigcirc, B_R \rangle) = \max(height(B_L), height(B_R)) + 1$
- $leaves(\Box) = 1$
- $leaves(\langle B_L, \bigcirc, B_R \rangle) = leaves(B_L) + leaves(B_R)$

Prove the following property for all binary trees $B \in \mathcal{B}$ by structural induction:

$$leaves(B) \leq 2^{height(B)-1}.$$