We want to prove that $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ is a model of

$$\varphi = \forall x \exists y\, ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y)))$$

with

$$\begin{aligned} U &= \{u_1, u_2\} \\ \mathrm{c}^{\mathcal{I}} &= u_1 \\ \mathrm{f}^{\mathcal{I}} &= \{u_1 \mapsto u_2, u_2 \mapsto u_1\} \\ \mathrm{P}^{\mathcal{I}} &= \{(u_2, u_2)\} \end{aligned}$$

Proof: Consider any variable assignment α. We first look at $\alpha_1 = \alpha[x := u_1][y := u_2]$.

Because $\mathrm{c}^{\mathcal{I}} = u_1$ and $\mathrm{f}(x)^{\mathcal{I},\alpha_1} = \mathrm{f}^{\mathcal{I}}(\alpha_1(x)) = \mathrm{f}^{\mathcal{I}}(u_1) = u_2 \neq u_1 = \mathrm{c}^{\mathcal{I}}$, we have

$$\mathcal{I}, \alpha_1 \not\models (\mathrm{f}(x) = c) \text{ or } \mathcal{I}, \alpha_1 \models \neg(\mathrm{f}(x) = c).$$

so we also know that

$$\mathcal{I}, \alpha_1 \models (\neg(\mathrm{f}(x) = c) \vee \mathrm{P}(x, y)) \text{ which means } \mathcal{I}, \alpha_1 \models ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y)).$$

Since there is a $u \in U$ (namely u_2) such that $\mathcal{I}, \alpha[x := u_1][y := u] \models ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y))$ we can use the definition of $\models$ for existentially quantified formulas to derive

$$\mathcal{I}, \alpha[x := u_1] \models \exists y\, ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y)) \quad (*)$$

We now look at $\alpha_2 = \alpha[x := u_2][y := u_2]$. Because $(\alpha_2(x), \alpha_2(y)) = (u_2, u_2) \in \mathrm{P}^{\mathcal{I}}$, we know that $\mathcal{I}, \alpha_2 \models \mathrm{P}(x, y)$ and (with the same reasoning as above), we conclude that

$$\mathcal{I}, \alpha_2 \models ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y)).$$

Since there is one $u \in U$ (namely u_2) such that $\mathcal{I}, \alpha[x := u_2][y := u] \models ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y))$ we can use the definition of $\models$ for existentially quantified formulas to derive

$$\mathcal{I}, \alpha[x := u_2] \models \exists y\, ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y)) \quad (**)$$

We have shown (in $(*)$ and $(**)$) that $\mathcal{I}, \alpha[x := u] \models \exists y\, ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y))$ holds for all $u \in U$, so we have

$$\mathcal{I}, \alpha \models \forall x \exists y\, ((\mathrm{f}(x) = c) \to \mathrm{P}(x, y))$$

which is what we wanted to show ($\mathcal{I}, \alpha \models \varphi$ for all variable assignments α).