Theory of Computer Science

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Spring Term 2015

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Computer Science

Exercise Sheet 1

Due: Wednesday, March 4, 2015

Note: Submissions that are exclusively created with LaTeX will receive a bonus mark. Please submit only the resulting PDF file (or a printout of this file).

Note: The goal of this exercise is to learn how to correctly express formal proofs. A formally correct proof consists of single steps where each step follows *immediately* from the previous steps or from the assumptions (for example when replacing a value by its definition). Please write down your proofs in detail and in a formal fashion. Examples can be fould in the lecture slides.

Exercise 1.1 (Direct Proof; 2 Points)

Prove the following statement for any sets A and B with a direct proof:

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Note: For such statements it is common to not specify any context for the complement, which implies that the statement is assumed to be true for any context. Since you will need to use the context for your proof, explicitely specify this assumption by defining the following in the beginning: "We consider the complement in the context of a set C where $A, B \subseteq C$."

Exercise 1.2 (Proof by Contradiction; 1,5 + 1,5 Points)

Prove the following statement by contradiction.

(a) For all $n \in \mathbb{N}_0$ the following holds: if n^2 is even, so is n.

(b) A directed *graph* $G = (V, E)$ consists of a finite set of *nodes* V and a set of *edges* $E \subseteq V^2$.

A *path* of length n from node u to node u' in G is a tuple $\langle v_1, \dots, v_n \rangle$ such that $v_1 = u, v_n = u'$ and for $1 \leq i < n$, $(v_i, v_{i+1}) \in E$.

Prove by contradiction: If $\langle v_1, \dots, v_n \rangle$ is a shortest path from v_1 to v_n in G, then the tuple $\langle v_i, \dots, v_j \rangle$ for any $i, j \in \mathbb{N}_0$ with $1 \leq i \leq j \leq n$ is a shortest path from v_i to v_j in G.

Exercise 1.3 (Mathematical Induction; 1 Point)

We want to prove that all giraffes are of equal height. Which line of the following proof by mathematical induction has an error? What exactly is the error on the formal level?

1. Since we do not know the exact number of giraffes we will prove that for any set of giraffes all giraffes in the set are of equal height.
2. We define the property $G(n)$: In any set of n giraffes, all giraffes are of equal height.
3. As the induction basis we show $G(1)$.
4. In a set of giraffes consisting of only one giraffe it is obvious that all giraffes in the set are of equal height. Thus $G(1)$ holds.
5. The induction hypothesis is: $G(i)$ holds for all $1 \leq i \leq n$.
6. We need to show $G(n+1)$ under the assumption of the induction hypothesis.

7. We consider a set of $n+1$ giraffes $M = \{g_1, \ldots, g_n, g_{n+1}\}$
8. Removing the first giraffe from the set M results in a set $M_1 = \{g_2, \ldots, g_{n+1}\}$ containing n giraffes ($|M_1| = n$).
9. According to the induction hypothesis $G(n)$ holds for M_1 and thus we can conclude that all giraffes in M_1 are of equal height.
10. Removing the last giraffe from the set M results in a set $M_2 = \{g_1, \ldots, g_n\}$ containing n giraffes ($|M_2| = n$).
11. According to the induction hypothesis $G(n)$ holds for M_2 and thus we can conclude that all giraffes in M_2 are of equal height.
12. Now we consider a giraffe g_M which is contained in both sets, for example $g_M = g_2$.
13. g_M has the same height as all giraffes from M_1 and also as all giraffes from M_2.
14. From this we conclude that all giraffes from $M_1 \cup M_2 = M$ must be of equal height, which means we have shown that $G(n+1)$ holds for M.
15. Since $G(1)$ holds, and, under hypothesis $G(n)$, $G(n+1)$ also holds for $n \geq 1$, we conclude that $G(n)$ holds for all $n \in \mathbb{N}$.
16. In particular $G(n)$ holds for the set of all giraffes, which means that all giraffes are of equal height.

Exercise 1.4 (Structural Induction; 2 + 2 Points)

(a) We inductively define a set of simple mathematical expressions which only utilize the following symbols: "0", "1", "·", "⊕", "(" and ")". The set $\mathcal{E}$ of *simple expressions* is inductively defined as follows:

- 0 and 1 are simple expressions.
- If x and y are simple expressions, $(x \cdot y)$ is also a simple expression.
- If x and y are simple expressions, $(x \oplus y)$ is also a simple expression.

Furthermore we define a function $f : \mathcal{E} \to \mathbb{N}_0$ as follows:

- $f(0) = 0$, $f(1) = 1$
- $f((x \cdot y)) = f(x) \cdot f(y)$
- $f((x \oplus y)) = (f(x) + f(y)) \mod 2$

Prove the following property for all simple expressions $x \in \mathcal{E}$ by structural induction:

$$f(x) \in \{0, 1\}.$$

(b) We define two functions over binary trees (as presented in the lecture): *height* $: \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to its height and *leaves* $: \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to the number of its leaves.

- $\mathit{height}(\square) = 1$
- $\mathit{height}((B_L, \bigcirc, B_R)) = \max(\mathit{height}(B_L), \mathit{height}(B_R)) + 1$
- $\mathit{leaves}(\square) = 1$
- $\mathit{leaves}((B_L, \bigcirc, B_R)) = \mathit{leaves}(B_L) + \mathit{leaves}(B_R)$

Prove the following property for all binary trees $B \in \mathcal{B}$ by structural induction:

$$\mathit{leaves}(B) \leq 2^{\mathit{height}(B)-1}.$$